Section 7.8
Repeated eigenvalues

Find the general solution of:

$$\vec{x}'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}(t)$$

We find the eigenvalues and eigenvectors.

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-1-\lambda) + 4 = 0$$

$$-3 - 3\lambda + \lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 1$$

We now proceed to compute the eigenvectors

$$\begin{pmatrix} 3-1 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_1 - 4x_2 = 0$$

$$x_1 - 2x_2 = 0$$

$$x_1 = 2x_2$$

The eigenspace is:

$$\left\{ \begin{pmatrix} 2r \\ r \end{pmatrix} = r \begin{pmatrix} 2 \\ 1 \end{pmatrix} : r \text{ is any number} \right\}$$

In particular:

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = (1)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We have:

$$\lambda = 1, \quad \vec{\xi} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We have seen in previous sections that:

$$\vec{x}^{(1)}(t) = e^{\lambda t}\vec{\xi} = e^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$ is a solution

to the system; but we only have one vector solution. We try to find $\vec{x}^{(2)}(t)$ of the form:

$$\vec{x}^{(2)}(t) = t\, e^{\lambda t}\,\vec{\xi}$$

We check:

$$\vec{x}^{(2)\prime}(t) = \left(e^{\lambda t} + t\lambda e^{\lambda t}\right)\vec{\xi} = e^{\lambda t}\vec{\xi} + t\lambda e^{\lambda t}\vec{\xi}$$

$$A\,\vec{x}^{(2)}(t) = A\left(t\,e^{\lambda t}\vec{\xi}\right) = t\,e^{\lambda t} A\vec{\xi}$$

$$= t\,e^{\lambda t}\lambda\vec{\xi}\ ; \quad \text{Since } A\vec{\xi} = \lambda\vec{\xi}$$

We want

$$\vec{x}^{(2)\prime}(t) = A\,\vec{x}^{(2)}(t),$$

but this is not true;

$$e^{\lambda t}\vec{\xi} + t\lambda \cancel{e^{\lambda t}}\,\vec{\xi} \neq t\,\cancel{e^{\lambda t}}\,\lambda\,\vec{\xi}$$

Thus, instead we look for $\vec{x}^{(2)}(t)$ of the form:

$$\vec{x}^{(2)}(t) = te^{\lambda t}\vec{\xi} + \vec{\eta}e^{\lambda t}$$

We need to find $\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ such that:

$$\vec{x}^{(2)\prime}(t) = A\,\vec{x}^{(2)}(t)$$

$$\vec{x}^{(2)\prime}(t) = \lambda te^{\lambda t}\vec{\xi} + e^{\lambda t}\vec{\xi} + \lambda e^{\lambda t}\vec{\eta}$$

$$A\,\vec{x}^{(2)}(t) = A\left(te^{\lambda t}\vec{\xi} + \vec{\eta}e^{\lambda t}\right)$$

$$= te^{\lambda t}\underline{A\vec{\xi}} + e^{\lambda t}A\vec{\eta}$$

$$= te^{\lambda t}\lambda\vec{\xi} + e^{\lambda t}A\vec{\eta}\text{; because } A\vec{\xi} = \lambda\vec{\xi}$$

We need $\vec{\eta}$ so that:

$$\cancel{\lambda te^{\lambda t}\vec{\xi}} + e^{\lambda t}\vec{\xi} + \lambda e^{\lambda t}\vec{\eta} = \cancel{te^{\lambda t}\lambda\vec{\xi}} + e^{\lambda t}A\vec{\eta}$$

$$e^{\lambda t}\left(\vec{\xi} + \lambda\vec{\eta}\right) = e^{\lambda t}A\vec{\eta}$$

$$A\vec{\eta} = \vec{\xi} + \lambda\vec{\eta} = \vec{\xi} + \lambda I\vec{\eta}$$

$$A\vec{\eta} - \lambda I\vec{\eta} = \vec{\xi}$$

$$\boxed{(A - \lambda I)\vec{\eta} = \vec{\xi}} \quad (1)$$

Using (1) in our example:

$$A - \lambda I = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$$

We solve:

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\left.\begin{array}{r} 2\eta_1 - 4\eta_2 = 2 \\ \eta_1 - 2\eta_2 = 1 \end{array}\right\} \Rightarrow \eta_1 = 1 + 2\eta_2$$

$$\vec{\eta} = \begin{pmatrix} 1+2\eta_2 \\ \eta_2 \end{pmatrix} = \eta_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = k \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where k is any real number.

Therefore:

$$\begin{aligned} \vec{x}^{(2)}(t) &= te^t \vec{\xi} + e^t \vec{\eta} \\ &= te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + ke^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

Since $\vec{x}^{(1)}(t) = e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ we can take $k = 0$; it is okay to pick $k \neq 0$, in this case the term $ke^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ would combine with the term $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in the general solution:

$$\boxed{\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]}$$

We double check that $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ are both solutions to the system:

$$\vec{x}\,'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}(t)$$

or

$$x_1'(t) = 3x_1(t) - 4x_2(t)$$
$$x_2'(t) = x_1(t) - x_2(t)$$

$\vec{x}^{(1)}(t) = e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, or $x_1(t) = 2e^t$, $x_2(t) = e^t$

$x_1'(t) = 2e^t = 3(2e^t) - 4(e^t) = 2e^t = 3x_1 - 4x_2$ ✓

$x_2'(t) = e^t = 2e^t - e^t = e^t = x_1 - x_2$ ✓

Also:

$\vec{x}^{(2)}(t) = te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

or $x_1(t) = 2te^t + e^t$, $x_2(t) = te^t$

$$\begin{aligned} x_1'(t) = 2e^t + 2te^t + e^t &= 3(2te^t + e^t) - 4(te^t) \\ &= 6te^t + 3e^t - 4te^t \\ &= 2te^t + 3e^t = 3x_1 - 4x_2 \ ✓ \end{aligned}$$

$x_2'(t) = te^t + e^t = 2te^t + e^t - te^t = te^t + e^t = x_1 - x_2$ ✓,

Hence $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ are solutions of

$$\vec{x}\,'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}(t).$$

Hence, the method we just developed is as follows:

$$\vec{x}'(t) = A\vec{x}(t)$$

λ is an eigenvalue that repeats.

Then:

$$A\vec{\xi} = \lambda\vec{\xi},$$ where $\vec{\xi}$ is an eigenvector for λ.

We have one solution:

$$\vec{x}^{(1)}(t) = e^{\lambda t}\vec{\xi}$$

We look for $\vec{\eta}$ such that:

$$\vec{x}^{(2)}(t) = te^{\lambda t}\vec{\xi} + e^{\lambda t}\vec{\eta}$$

is a solution to the system. We find $\vec{\eta}$ by solving:

$$(\vec{A} - \lambda I)\vec{\eta} = \vec{\xi}$$

The general solution is:

$$\begin{aligned}\vec{x}(t) &= c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) \\ &= c_1 e^{\lambda t}\vec{\xi} + c_2\left[te^{\lambda t}\vec{\xi} + e^{\lambda t}\vec{\eta}\right]\end{aligned}$$