Lecture 20

Another proof of the $\mathcal{H}^{n-1}$-rectifiability of $\partial^* E$, using Whitney's extension theorem, now follows:

Theorem 2: (De Giorgi's structure theorem). If E is a set of locally finite perimeter in $\mathbb{R}^n$, then:

$$\mu_E = \nu_E \, \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$$

Moreover, there exist countably many C^1-hypersurfaces $M_i \subset \mathbb{R}^n$, compact sets:

$$K_i \subset M_i,$$

and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$, such that:

$$\partial^* E = F \cup \left(\overset{\infty}{\underset{i=1}{\cup}} K_i \right)$$

and, for every $x \in K_i$, $\nu_E(x)^\perp = T_x M_i$, the tangent space to M_i at x.

Remark: Since

$$\int_E \nabla \varphi = \int_{\mathbb{R}^n} \varphi \, d\mu_E \qquad \forall \varphi \in C_c^1(\mathbb{R}^n)$$

then $\mu_E = \nu_E \, \mathcal{H}^{n-1} \llcorner \partial^* E$ and hence

$$\int_E \nabla \varphi = \int_{\partial^* E} \varphi \, \nu_E \, d\mathcal{H}^{n-1} \qquad \forall \varphi \in C_c^1(\mathbb{R}^n)$$

We need Whitney's extension theorem
to prove theorem 2:

Theorem (Whitney's extension theorem). Let
$C \subset \mathbb{R}^n$ closed, $u: C \to \mathbb{R}$ continuous, $T: C \to \mathbb{R}^n$ continuous.

Then $\exists \, v \in C^1(\mathbb{R}^n)$ s.t:

$$u = v \text{ on } C, \qquad T = \nabla v \text{ on } C$$

if and only if:

$$\lim_{\delta \to 0^+} \sup \left\{ \frac{|u(y) - u(x) - T(x) \cdot (y-x)|}{|x-y|} : 0 < |x-y| \leq \delta, \, x, y \in K \right\} = 0,$$
$$\forall K \subset C.$$

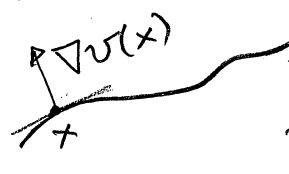Corollary: If $K \subset \mathbb{R}^n$ is a compact set, $T: K \to \mathbb{R}^n$
is continuous, and

$$\lim_{\delta \to 0^+} \sup \left\{ \frac{T(x) \cdot (y-x)}{|x-y|} : 0 < |x-y| \leq \delta, \, x, y \in K \right\} = 0$$

then
$\exists \, v \in C^1(\mathbb{R}^n)$ such that:
$$K \subset \{ x \in \mathbb{R}^n : v(x) = 0 \}, \quad T = \nabla v \text{ on } K.$$

In particular, if $T \neq 0$ on K, then K is
contained in a C^1-hypersurface.

$K \subset v^{-1}(0)$ ∇v is orthogonal
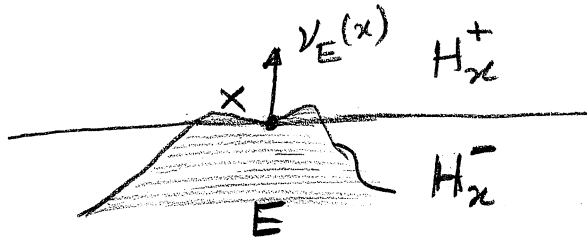$v \in C^1(\mathbb{R}^n$ to level surface $v^{-1}(0)$.

Proof of Theorem 2 :

Let $x \in \partial^* E$.

Define :

$$H_x^- = \{ y \in \mathbb{R}^n : (y-x) \cdot \nu_E(x) \leq 0 \}, \quad H_x^+ = \{ y \in \mathbb{R}^n : (y-x) \cdot \nu_E(x) \geq 0 \}$$

Since $E_{x,r} \xrightarrow{\text{loc}} H_x^-$, as $r \to 0$ and $\dfrac{|E \cap B(x,r)|}{\omega_n r^n} \to \dfrac{1}{2}$

as $r \to 0$ we have:

$$\lim_{r \to 0^+} \frac{|B(x,r) \cap H_x^+ \cap E|}{\omega_n r^n} = 0, \quad \lim_{r \to 0^+} \frac{|B(x,r) \cap H_x^- \cap E|}{\omega_n r^n} = \frac{1}{2} \quad (\ast)$$

WLOG E is of finite perimeter (Otherwise we write $E = \bigcup_{K=1}^{\infty} (E \cap B_K)$, where each $E \cap B_K$ is of finite perimeter because $P(E \cap B_R) = P(E; B_R) + \mathcal{H}^{n-1}(E \cap \partial B_R)$ for a.e. R .).

Since $|\mu_E|(\partial^* E) = P(E) < \infty$, We can apply Egoroff's Theorem to the relations in $(\ast)$. Thus, $\exists F_i$, for every i, a $|\mu_E|$-measurable set such that:

$$F_i \subset \partial^* E, \quad |\mu_E|(\partial^* E \setminus F_i) < \frac{1}{2^i}, \quad \text{and the limits}$$

in $(\ast)$ are uniform.

Further, by Lusin's theorem $\exists C_{i,j}$ $j=1,2,-$ (20.4)
compact set such that:

$$\boxed{|\mu_E|(F_i \setminus C_{i,j}) < \frac{1}{2^j} \quad, \quad \nu_E|_{C_{i,j}} \text{ is continuous.}}$$

With this:

$$|\mu_E|\left(F_i \setminus \bigcup_{j=1}^{\infty} C_{i,j}\right) = |\mu_E|\left(\bigcap_{j=1}^{\infty} F_i \setminus C_{i,j}\right)$$

$$\leq |\mu_E|\left(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} F_i \setminus C_{i,j}\right) \quad \text{Since } \bigcap_{j=1}^{\infty} F_i \setminus C_{i,j}$$

is contained in

$$= 0; \quad \text{by Borel} \quad \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} F_i \setminus C_{i,j}$$
Cantelli Lemma.

In the same way:

$$|\mu_E|\left(\partial^* E \setminus \bigcup_{i=1}^{\infty} F_i\right) = |\mu_E|\left(\bigcap_{i=1}^{\infty} \partial^* E \setminus F_i\right)$$

$$\leq |\mu_E|\left(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \partial^* E \setminus F_i\right)$$

$$= 0.$$

Therefore:

$$|\mu_E|\left(\partial^* E \setminus \bigcup_{i,j=1}^{\infty} C_{i,j}\right) = 0$$

We can rename the compact sets $C_{i,j}$ as $K_1, K_2, \cdots$.
We have then:

(i) $|\mu_E|(F) = 0$, $F = \partial^* E \setminus \bigcup_{i=1}^{\infty} K_i$

(ii) ν_E is continuous on K_i, $\forall i$

(iii) The limits in ($*$) are uniform.

We consider only one K_i, and we show that
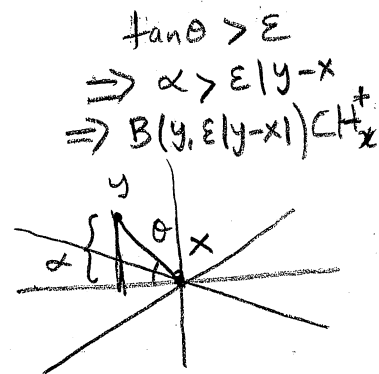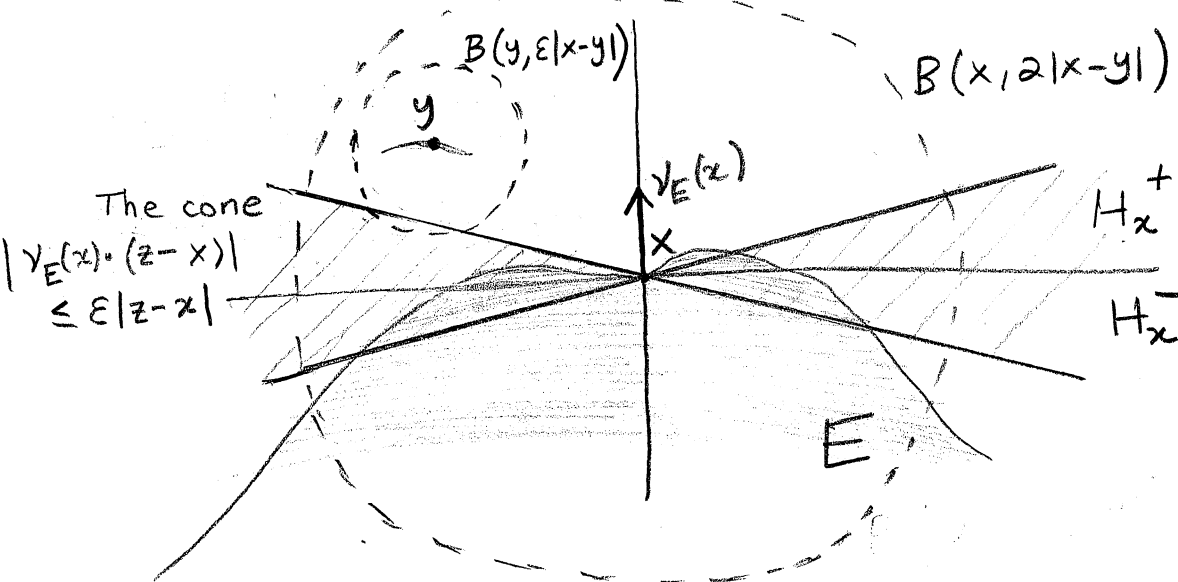(ν_E, K_i) satisfies the hypothesis of the Corollary

$\underline{\text{Claim}}$: $\forall \, \varepsilon \in (0,1) \, \exists \, \delta(\varepsilon, i, n)$ s.t:

$$\begin{cases} |x-y| \leq \delta \\ x, y \in K_i \end{cases} \implies |\nu_E(x) \cdot (y-x)| \leq \varepsilon |x-y|$$

We proceed by contradiction. Then $\exists \, \varepsilon > 0$ s.t. $\forall \delta > 0$:

$$\begin{aligned} |x-y| \leq \delta \\ x, y \in K_i \end{aligned} \implies |\nu_E(x) \cdot (y-x)| > \varepsilon |x-y|.$$

Thus, if $|y-x| \leq \delta$ then y is outside the cone $\{z : |\nu_E(x) \cdot (z-x)| \leq \varepsilon |z-x| \}$.

$$|\nu_E(x) \cdot (y-x)| > \varepsilon |x-y| \implies \varepsilon |x-y| < |y-x|$$

$$\implies B(y, \varepsilon|x-y|) \subset B(x, 2|x-y|) \cap H_x^+$$

We find a contradiction, using (*):

$$\frac{1}{2} \xleftarrow[|x-y| \to 0]{\text{unit.}} \frac{|B(y, \varepsilon|x-y|) \cap H_y^- \cap E|}{\varepsilon^n |x-y|^n} \leq \frac{|B(y, \varepsilon|x-y|) \cap E|}{\varepsilon^n |x-y|^n}$$

$$\therefore \forall \gamma > 0 \, \exists \, \delta(\gamma) \text{ s.t.} \quad |B(y, \varepsilon|x-y|) \cap E| \geq \varepsilon^n |x-y|^n \left(\tfrac{1}{2} - \gamma\right) \quad \substack{\forall y, \\ |y-x| \leq \delta}$$

$$|B(y, \varepsilon|x-y|) \cap E| \leq |B(x, 2|x-y|) \cap E \cap H_x^+|$$

since $B(y, \varepsilon|x-y|) \subset H_x^+$

And

$$\frac{|B(x, 2|x-y|) \cap E \cap H_x^+|}{2^n|x-y|^n} \xrightarrow[\text{unif.}]{|x-y| \to 0} 0$$

Thus, $\forall \gamma > 0 \; \exists \delta(\gamma)$ s.t.

$$|B(x, 2|x-y|) \cap E \cap H_x^+| \leq \gamma(\delta) \, 2^n |x-y|^n, \quad \forall y, \; |y-x| \leq \delta$$

It can not be "big and small" at the same time. Indeed, for γ very small:
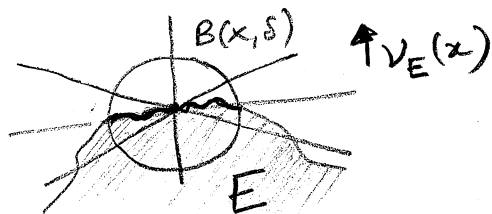
$$\varepsilon^n |x-y|^n \left(\tfrac{1}{2} - \gamma\right) \leq \gamma \, 2^n |x-y|^n$$

$$\underbrace{\tfrac{1}{2} - \gamma}_{\text{almost } \frac{1}{2}} \leq \gamma \underbrace{\left(\tfrac{2}{\varepsilon}\right)^n}_{\text{almost } 0} \implies \text{We have a contradiction.}$$

We have proved our claim, which says that $\forall \varepsilon > 0$, $\exists \delta$ s.t. $B(x, \delta) \cap K_i$ is contained in the cone:

$$\{z : \nu_E(x) \circ (z - x)| \leq \varepsilon |z - x|\}$$

Note: If y is on the other side of the cone in the picture in Page 20.5, proceeding in the same way we get again a contradiction. ∎

The claim we just proved says that:

$$\lim_{|x-y| \to 0} \frac{|\nu_E(x) \cdot (y-x)|}{|x-y|} = 0, \quad \text{uniformly on } x, y \in K_i$$

which is the hypothesis of the Corollary of the Whitney's extension theorem. Therefore $\exists \nu_i \in C'(\mathbb{R}^n)$ such that:

$$\boxed{K_i \subset \nu_i^{-1}(0), \quad \nabla \nu_i = \nu_E \text{ on } K_i.}$$

Since $\nabla \nu_i(x) = \nu_E(x) \neq 0$ on K_i then

$$\boxed{K_i \text{ is contained on a } C'-\text{hypersurface } M_i.}$$

Note: Since $\nabla \nu_i \neq 0$ on K_i, then by the continuity of $\nabla \nu_i$ ($\nu_i \in C'(\mathbb{R}^n)$) we have:

$$\overline{K_i} \subset \nu_i^{-1}(0), \quad \nabla \nu_i \neq 0 \text{ on } \overline{K_i}$$

Also, there exists an open set $A \subset \mathbb{R}^n$, $\overline{K_i} \subset A$, such that $A \cap \nu_i^{-1}(0)$ is a $C'-$hypersurface. Thus,

$$M_i := A \cap \nu_i^{-1}(0).$$

Recall the following:

Definition: A subset S of $\mathbb{R}^n$ is called a hypersurface of class C^k, $1 \leq k \leq \infty$, if for every $x_0 \in S$ there is an open set $V \subset \mathbb{R}^n$ containing x_0 and a real valued function $\phi \in C^k(V)$ such that $\nabla \phi$ is nonvanishing on $S \cap V$ and $S \cap V = \{x \in V : \phi(x) = 0\}$. In this case, by the implicit function theorem we can solve the equation $\phi(x) = 0$ near x_0 for some coordinate x_i (for convinience, say $i = n$) to obtain:

$$x_n = \Psi(x_1, \dots, x_{n-1}), \quad \Psi \in C^k$$

To finish the proof of Theorem 2, $\quad$ (20.8)
we need to show that:

$$|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E .$$

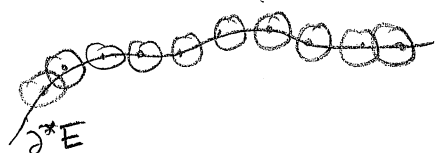We first show that $\forall G \subset \mathbb{R}^n$ Borel:

$$\boxed{\mathcal{H}^{n-1}(G \cap \partial^* E) \leq \tilde{c}(n) |\mu_E|(G \cap \partial^* E)} \quad (1)$$

Let A open set with:

$$G \cap \partial^* E \subset A .$$

Let:
$$\mathcal{F} = \left\{ B(x,r) \subset A : x \in G \cap \partial^* E, 0 < r < \delta, |\mu_E|(B(x,r)) \geq \frac{\omega_{n-1} r^{n-1}}{2} \right\}$$

$\partial^* E$

Recall that
$$\frac{|\mu_E|(B(x,r))}{\omega_{n-1} r^{n-1}} \to 1$$

Besicovitch's Covering theorem $\Rightarrow$:

$$\exists \{\mathcal{F}_i\}_{i=1}^{c(n)} \quad \mathcal{F}_i = \{B_{i,1}, B_{i,2}, B_{i,3}, \dots\}$$
$$= \{B_{i,j}\}_{j=1}^{\infty} , \text{ disjoint} .$$

$$G \cap \partial^* E \subset \bigcup_{i=1}^{c(n)} \mathcal{F}_i$$

$$\mathcal{H}_\delta^{n-1}(G \cap \partial^* E) \leq \sum_{i=1}^{c(n)} \sum_{B(x,r) \in \mathcal{F}_i} \omega_{n-1} r^{n-1}$$

$$\leq 2 \sum_{i=1}^{c(n)} \sum_{B(x,r) \in \mathcal{F}_i} |\mu_E|(B(x,r))$$

$$\leq 2c(n) |\mu_E|(A), \text{ each } \mathcal{F}_i \text{ is disjoint}$$

Let $\delta \to 0^+$:

$$\mathcal{H}^{n-1}(G \cap \partial^* E) \leq \tilde{c}(n) |\mu_E|(A), \quad A \text{ arbitrary} \Rightarrow$$
$$\mathcal{H}^{n-1}(G \cap \partial^* E) \leq |\mu_E|(G \cap \partial^* E).$$

From (1):

$$\mathcal{H}^{n-1} \llcorner \partial^* E \text{ is a Radon measure.}$$

By exercise 2.6, it is enough to show:

$$\mathcal{H}^{n-1} \llcorner \partial^* E = |\mu_E| \quad \text{on } \mathcal{B}(\mathbb{R}^n).$$

We have proved:

$$\partial^* E = F \cup \left(\bigcup_{i=1}^{\infty} K_i \right), \quad |\mu_E|(F) = 0.$$

We only need to show:

$$\mathcal{H}^{n-1} \llcorner K_i = |\mu_E| \llcorner K_i \quad \text{on } \mathcal{B}(\mathbb{R}^n), \forall i.$$

By Lebesgue – Besicovitch differentiation theorem:

$$\mathcal{H}^{n-1} \llcorner M_i = \theta |\mu_E| + \mathcal{H}^{n-1} \llcorner Y_i, \quad Y_i \subset M_i, \quad |\mu_E|(Y_i) = 0$$

$$\theta(x) = \lim_{r \to 0^+} \frac{\mathcal{H}^{n-1}(M_i \cap B(x,r))}{|\mu_E|(B(x,r))}, \quad |\mu_E| - \text{a.e. } x \in \partial^* E$$

M_i is C^1-hypersurface $\Rightarrow \lim\limits_{r \to 0^+} \dfrac{\mathcal{H}^{n-1}(M_i \cap B(x,r))}{\omega_n r^{n-1}} = 1, \quad x \in M_i$

$x \in \partial^* E \Rightarrow \lim\limits_{r \to 0} \dfrac{|\mu_E|(B(x,r))}{\omega_n r^{n-1}} = 1$

$$\therefore \quad \theta = 1 \quad \text{on } K_i$$

G Borel, $G \subset K_i \Rightarrow \underline{\mathcal{H}^{n-1}(G)} = \mathcal{H}^{n-1}(G \cap K_i) \underline{= |\mu_E|(G)}$

because $\mathcal{H}^{n-1}(G \cap Y_i) \le \mathcal{H}^{n-1}(Y_i \cap \partial^* E) \le \tilde{c}(n) |\mu_E|(Y_i) = 0.$

Thus:

$$\mathcal{H}^{n-1} \llcorner K_i = |\mu_E| \llcorner K_i \quad \text{on } \mathcal{B}(\mathbb{R}^n).$$