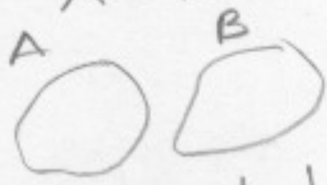
Connected Sets

<u>Def</u>: Let X be a metric space.
$A, B \subset X$ are said to be separated if:

$$A \cap \bar{B} = \emptyset \text{ and } \bar{A} \cap B = \emptyset$$

(recall, $\bar{E} = E \cup E'$, is the closure of E).

<u>Ex</u>: $X = \mathbb{R}^2$



A, B are separated

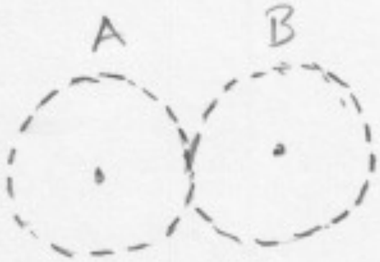
$E = A \cup B$ is not connected

$A = \bar{B}_1(0,0)$, $B = B_1(2,0)$

A, B are not separated since

$A \cap \bar{B} \neq \emptyset$

$E = A \cup B$ is connected



$A = B_1(0,0)$, $B = B_1(2,0)$

A, B are separated since

$\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$

$E = A \cup B$ is not connected.

(Recall notation: If $X = \mathbb{R}^k$, $B_r(\vec{x})$ is the open ball of radius r centered a $\vec{x}$, which is an open neighborhood of $\vec{x}$. All the neighborhoods in $\mathbb{R}^k$ are open balls).

Definition $E \subset X$ is connected if E is not a union of two non-empty separated sets.

That is:
If E is the union of two non-empty separated sets, then E is not connected.

We have the following:

Theorem: $E \subset \mathbb{R}^1$ is connected if and only if it has the following property:

If $x, y \in E$ and $x < z < y \Rightarrow z \in E$ $(*)$

Proof:
(a) We prove first the implication:
If $E \subset \mathbb{R}$ is connected $\Rightarrow$ If $x, y \in E$, and $x < z < y$, z must belong to E.

We proceed by contradiction. Assume $\exists x, y \in E$ and $z \in (x, y)$, s.t. $z \notin E$. Then:

$$E = [\underbrace{E \cap (-\infty, z)}_{A_z}] \cup [\underbrace{E \cap (z, \infty)}_{B_z}]$$

$$x \in A_z, \; y \in B_z \Rightarrow A, B \neq \emptyset$$

Since $(-\infty, z)$ and (z, ∞) are separated then A_z and B_z are separated. Since E is the union of non-empty separated sets, then E is not connected, which is a contradiction.

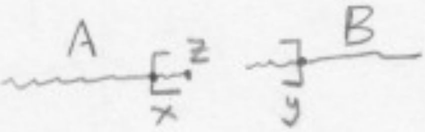
(b) We now prove the other implication:

Assume that E satisfies (*). We need to show that E is connected. Suppose E is not connected. Then it is the union of separated sets:

$$E = A \cup B, \quad A, B \neq 0, \quad \overline{A} \cap B = \emptyset, \ \overline{B} \cap A = \emptyset$$

Pick $x \in A$, $y \in B$ and assume (without loss of generality) that $x < y$. Define:

$$(**) \qquad z = \sup (A \cap [x, y])$$



(Recall Theorem 2.28: Let $E \neq \emptyset$, $E \subset \mathbb{R}^1$, E bounded above. If $y = \sup E$, then $y \in \overline{E}$). With $E = A \cap [x, y]$, from Thm 2.28 we deduce that $z \in \overline{A}$. Since $\overline{A} \cap B = \emptyset$ then $z \notin B$ and hence $z \neq y$. Since $z \in [x, y]$ we conclude $z < y$. Since $z = \sup (A \cap [x, y])$ and $x \in A \Rightarrow x \leq z$. Thus:

$$x \leq z < y$$

We have two possibilities: $z \in A$ or $z \notin A$.

<u>Case 1</u>: If <u>$z \notin A$</u> then, since $x \in A$, from (**) we get:

$$x < z < y, \qquad (1)$$

but $z \notin E$ because $z \notin A$ and $z \notin B$. (1) contradicts (*).

<u>Case 2</u> : If $z \in A$, since $A \cap \overline{B} = \emptyset$ we obtain that $z \notin \overline{B}$ and hence $\exists z_1$ s.t. ;

$$z < z_1 < y, \quad z_1 \notin B, \quad (2)$$

Indeed, $\overline{B}$ is closed and $z \in (\overline{B})^c$, which is open, so there exists an open neighborhood $(z-\varepsilon, z+\varepsilon) \subset (\overline{B})^c$. We can then choose such z_1.

From (**) and (2):

$$x \leq z < z_1 < y, \text{ i.e., } x < z_1 < y,$$

but $z_1 \notin E$ because $z_1 > z = \sup(A \cap [x,y])$ and hence $z_1 \notin A$. Also, (2) $\Rightarrow$ $z_1 \notin B$. Therefore, $z_1 \notin A \cup B = E$. Since:

$$x < z_1 < y, \; z_1 \notin E, \quad (3)$$

Since (3) is a contradiction to (*), we conclude that E is connected. ∎