Chapter 3

Numerical Sequences and Series.

Def: A sequence $\{p_n\}$ in a metric space X is said to converge if there exists $p \in X$ such that:

For all $\varepsilon > 0$, there exists $N > 0$ such that:

$$d(p_n, p) < \varepsilon, \text{ for all } n \geq N.$$

We can rewrite the previous statement as follows (using our notation):

$$\forall \varepsilon > 0, \exists N > 0 \text{ s.t. } d(p_n, p) < \varepsilon, \forall n \geq N$$

Remark: N depends on ε. Sometimes we write $N(\varepsilon)$ to remark the dependance on ε.

Notation: If p_n converges to p we write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$

If $\{p_n\}$ does not converge, it is said to diverge.

Given a sequence $\{p_n\}$, we can associate to $\{p_n\}$ the function $f: J \to \{p_n\}$ as:

$$f(n) = p_n, \quad n = 1, 2, \ldots$$

Since elements of $\{p_n\}$ can repeat, the range of f is a countable set, or a finite set.

The sequence $\{p_n\}$ is bounded if the range of the sequence (i.e., the range of f) is a bounded set.

Ex : $X = \mathbb{R}$, $s_n = \frac{1}{n}$, $\lim_{n\to\infty} s_n = 0$, the range of the sequence is infinite, the sequence is bounded.

Ex: $X = \mathbb{R}$, $s_n = n^2$, $\{s_n\}$ is unbounded, it is divergent, it has infinite range.

Ex: $X = \mathbb{R}$, $s_n = 1$, $n = 1, 2, \ldots$, $\{s_n\}$ converges to 1, is bounded, and has finite range.

Ex: $X = \mathbb{R}$, $s_n = \frac{3n+2}{n+1}$. Show using the definition of convergence that:

$$\lim_{n\to\infty} s_n = 3$$

Let $\varepsilon > 0$. We need to show that $\exists N$ s.t:

$$\left| \frac{3n+2}{n+1} - 3 \right| < \varepsilon, \quad \forall n \geq N$$

we estimate:

$$\left| \frac{3n+2}{n+1} - 3 \right| = \left| \frac{3n+2-3n-3}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} < \frac{1}{n} \quad (1)$$

Hence, since $\frac{1}{n} \to 0$, $\exists N$ s.t.

$$\frac{1}{n} < \varepsilon \quad \forall n \geq N$$

From (1):

$$d(s_n, 3) = \left| \frac{3n+2}{n+1} - 3 \right| = \frac{1}{n+1} < \frac{1}{n} < \varepsilon, \quad \forall n \geq N$$

$$\Rightarrow \quad d(s_n, 3) < \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \quad s_n \to 3. \quad \blacksquare$$

<u>Ex</u>: $X = \mathbb{R}$, the sequence $0, 2, 0, 2, 0, 2, \dots, 0, 2, \dots$ is divergent

<u>Remark</u>: The definition of "convergence sequence" depends on X. For example:

(1) $X = \mathbb{R}^1$, with $d(x,y) = |x - y|$.
$\frac{1}{n} \to 0$ in X

(2) $X = (0, \infty)$ with $d(x,y) = |x-y|$. In this metric space, $\{\frac{1}{n}\}$ is divergent, because $0 \notin X$.

<u>Theorem 3.2</u> : Let $\{p_n\}$ be a sequence in a metric space X.

(a) $p_n \to p$, $p \in X$ if and only if every neighborhood of p contains all but finitely many of the terms of $\{p_n\}$

(b) If $p \in X$, $p' \in X$, and if $p_n \to p$ and $p_n \to p'$, then $p = p'$ (i.e; limits are unique)

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded

(d) If $E \subset X$, and $p \in E'$, then there exists a sequence $\{p_n\}$ in E such that $p_n \to p$.

<u>Proof</u>:

(a) $\Longrightarrow$

Suppose $p_n \to p$.
Let $N_r(p)$ be any neighborhood of p.
Let $\varepsilon := r$
Since $p_n \to p$, then $\exists N$, s.t.,

$$d(p_n, p) < \varepsilon, \quad \forall n \geq N$$

Since $\varepsilon = r$ it follows that $p_n \in N_r(p)$, $\forall n \geq N$, and hence $N_r(p)$ contains all but finitely many of the terms of $\{p_n\}$.

(b) $\Longleftarrow$

Suppose now that every $N_r(p)$, $r > 0$, contains all but finitely many of the terms in $\{p_n\}$.

- Let $\varepsilon > 0$.

Consider $N_\varepsilon(p)$. By assumption, $\exists N$ such that $p_n \in N_\varepsilon(p)$, if $n \geq N$. This means that $d(p_n, p) < \varepsilon$, $\forall n \geq N$. That is, $p_n \to p$.

(b) Let $\varepsilon > 0$.

Since $p_n \to p$, and $p_n \to p'$, $\exists N$, $\exists N'$ s.t.:

$$d(p_n, p) < \frac{\varepsilon}{2} \quad \forall n \geq N$$
$$d(p_n, p') < \frac{\varepsilon}{2} \quad \forall n \geq N'$$

Let $M := \max\{N, N'\}$. Thus,

$$d(p, p') \leq d(p, p_M) + d(p_M, p')$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\Rightarrow \quad \underline{d(p, p') < \varepsilon}$$

Since ε is arbitrary we conclude that

$$d(p, p') = 0,$$

that is,

$$p = p'.$$

(c) Suppose $p_n \to p$.

Let $\varepsilon = 1$

By definition of convergence, for this choice of ε, there exists N such that:

$$d(p_n, p) < 1, \quad \forall n \geq N,$$

which implies that:

$$p_n \in N_1(p) \quad \forall n \geq N$$

Let:

$$r := \max \{1, d(p_1, p), d(p_2, p), \dots, d(p_{N-1}, p)\}$$

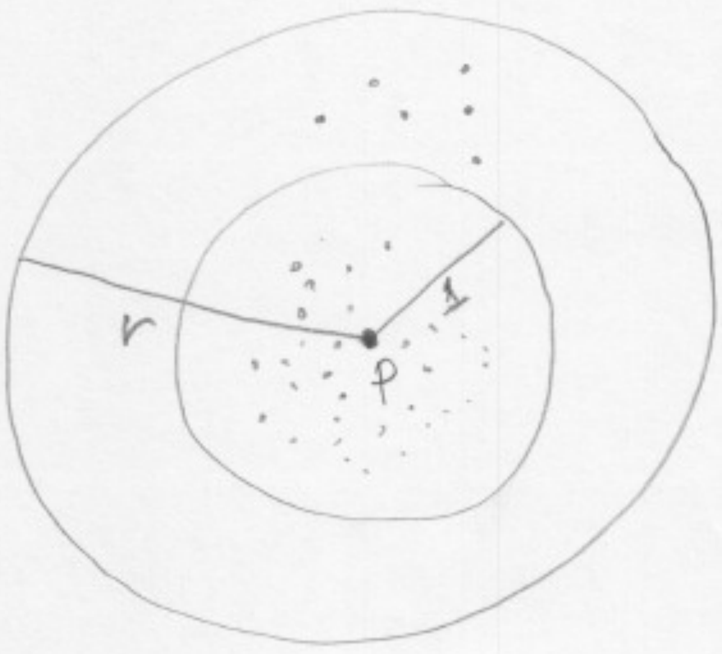
Hence:

$$d(p_n, p) \leq r \quad , \; n = 1, 2, \dots,$$

that is,

$$p_n \in N_r(p), \quad n = 1, 2, \dots$$

$\Rightarrow \{p_n\}$ is bounded. ∎



(d) Let $p \in E'$. Hence, for every n, the neighborhood $N_{1/n}(p)$ contains a point, say p_n, $p_n \neq p$, $p_n \in E$. We have:

$$d(p_n, p) < \frac{1}{n}$$

In this way we construct a sequence $\{p_n\}$ in E such that $p_n \to p$. Indeed, let $\varepsilon > 0$, then $\exists N$ such that:

$$\frac{1}{n} < \varepsilon \quad \forall n \geq N.$$

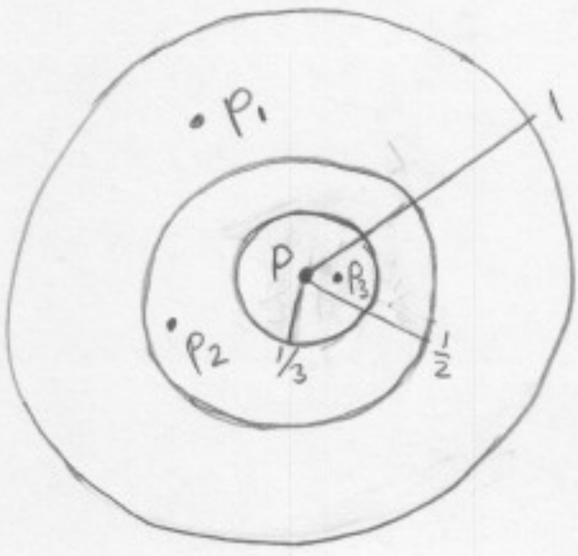
Therefore:

$$d(p_n, p) < \frac{1}{n} < \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \underline{d(p_n, p) < \varepsilon \ \forall n \geq N} \qquad (1)$$

The validity of (1) is precisely the definition of convergence. We conclude:

$$p_n \to p.$$



- Theorem 3.3 - 3.4 : Suppose $\{\vec{x}_n\}, \{\vec{y}_n\}$ are sequences in $\mathbb{R}^k$, and $\{p_n\}, \{q_n\}$ are sequences in $\mathbb{R}$. Then, if $p_n \to p$, $q_n \to q$, $\vec{x}_n \to \vec{x}$, $\vec{y}_n \to \vec{y}$, the following is true:

(a) $p_n + q_n \to p + q$

(b) $cp_n \to cp$, $c + p_n \to c + p$, for any number c

(c) $p_n q_n \to pq$

(d) $\frac{1}{p_n} \to \frac{1}{p}$, if $p_n \neq 0$, $n = 1, 2, \ldots$, $p \neq 0$

(e) If $\vec{x}_n = (x_{1,n}, x_{2,n}, \ldots, x_{k,n})$, then: $\vec{x}_n \to \vec{x}$, $\vec{x} = (\alpha_1, \ldots, \alpha_k)$ if and only if:

$$\alpha_{j,n} \to \alpha_j, \quad 1 \leq j \leq k$$

(f) $\vec{x}_n + \vec{y}_n \to \vec{x} + \vec{y}$

(g) $c\vec{x}_n \to c\vec{x}$, for any number c

(h) $\vec{x}_n \cdot \vec{y}_n \to \vec{x} \cdot \vec{y}$