Subsequences

Def.: Let $\{p_n\}$ be a sequence in X. If $n_1 < n_2 < n_3 < \dots$ is a sequence of positive integers, then:

$p_{n_1}, p_{n_2}, p_{n_3}, \dots$ is a subsequence of $\{p_n\}$.

If $\{p_{n_i}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

Ex: $s_n = \frac{1}{n}$, $n = 1, 2, \dots$

$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots, \frac{1}{2i+1}, \dots$ is a subsequence of $\{s_n\}$.

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots, \frac{1}{2i}, \dots$, is a subsequence of $\{s_n\}$.

Theorem 3.6:

(a) If $\{p_n\}$ is a sequence in a compact metric space X, then there is a subsequence $\{p_{n_i}\}$ and $p \in X$ such that $p_{n_i} \to p$ as $i \to \infty$

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence (This is also called the Bolzano-Weierstrass theorem).

Proof: We have $\{p_n\}$ a sequence in X

$$f(n) = p_n \ , \ n = 1, 2, \ldots$$

Let $E = \text{Range}(f)$.

Case 1: If E is a finite set, then $\exists\, p \in E$ and $n_1 < n_2 < n_3 \ldots$ such that

$$p_{n_1} = p_{n_2} = \cdots = p.$$

Clearly, $p_{n_i} \to p$

Case 2: If E is an infinite set, then Theorem 2.37 implies that E has a limit point $p \in X$.

For each $i = 1, 2, \ldots$, since p is a limit point, the neighborhood $N_{\frac{1}{i}}(p)$ contains a point, say p_{n_i}, $p_{n_i} \neq p$, $n_i > n_{i-1}$, $p_{n_i} \notin E$.

Clearly,

$$d(p_{n_i}, p) < \frac{1}{i}, \ i = 1, \ldots$$

Hence $p_{n_i} \longrightarrow p$. ∎

(b) Let $\{p_n\}$ be a bounded sequence in $\mathbb{R}^k$ and let E be the range of $\{p_n\}$. Since E is bounded, $E \subset I$, for some k-cell. Since I is compact, from (a), there exists a subsequence $\{p_{n_i}\}$ and $p \in \mathbb{R}^k$ s.t $p_{n_i} \to p$ as $i \to \infty$ ∎

- Theorem 3.7: Let X be a metric space and $\{p_n\}$ a sequence in X. Define:

$$E^* = \{q \in X : p_{n_i} \to q, \text{ for some subsequence } \{p_{n_i}\}\}$$

That is, E^* is the set of all subsequential limits of $\{p_n\}$.

Then E^* is a closed subset of X.

Proof: Let $q \in (E^*)'$.

We need to show that $q \in E^*$

Choose n_1 such that $p_{n_1} \neq q$ (if such n_1 does not exist then $E^* = \{q\}$ and clearly E^* is closed).

Let:

$$\delta := d(q, p_{n_1})$$

For each $i = 2, 3, 4, \ldots$, choose n_i as follows:

$$\begin{cases} q \in (E^*)' \Rightarrow \exists x \in E^* \text{ such that } d(q,x) < \frac{\delta}{2^i}, \\ \text{Now, } x \in E^* \Rightarrow \exists n_i \text{ such that } d(x, p_{n_i}) < \frac{\delta}{2^i} \\ \text{Choose } n_i > n_{i-1} \end{cases}$$

We estimate:

$$d(p_{n_i}, q) \leq d(p_{n_i}, x) + d(x, q)$$
$$< \frac{\delta}{2^i} + \frac{\delta}{2^i} = \frac{2\delta}{2^i} = \frac{\delta}{2^{i-1}}$$

We have constructed a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $d(p_{n_i}, q) < \frac{\delta}{2^{i-1}} < \varepsilon$, for i large enough.

Clearly, $p_{n_i} \to q$ as $i \to \infty$ and hence $q \in E^*$. ∎

The extended real number system.

<u>Definition</u>: The <u>extended</u> real number system consists of the real numbers $\mathbb{R}$ and two <u>symbols</u>, $+\infty$ and $-\infty$. We define

$$-\infty < x < \infty , \quad \forall x \in \mathbb{R}$$

Clearly, $+\infty$ is an upper bound of every subset of the extended real number system. Also, every non-empty subset has a least upper bound.

If, for example, E is a non-empty set of real numbers which is not bounded above in $\mathbb{R}$, then $\sup E = +\infty$ in the extended real number system.

The same remarks apply to lower bounds.

<u>Remark</u>: The extended real number system $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ is not a metric space. The definition of convergence in the metric space $X = \mathbb{R}$ (Definition 3.1) is not changed.

lim inf and lim sup

We will work with the extended real system, but we keep in mind the following:

Remark: The metric space $X=\mathbb{R}$ does not contain $\pm\infty$. The definition of convergence in $\mathbb{R}$ requires that the sequence and the limit belong to $\mathbb{R}$:

$$s_n \to s, \quad \underline{s\in\mathbb{R}} \quad \text{means:}$$

$$\forall \varepsilon>0, \ \exists N \text{ s.t. } |s_n - s| < \varepsilon, \ \forall n \geq N. \quad (1)$$

In (1), s can not be $+\infty$ or $-\infty$.

The sequence $\{n^2\}$ is not convergent in $\mathbb{R}$. This sequence is divergent. Indeed it diverges to $+\infty$. However, in order to simplifly the definitions of lim inf and lim sup, we will abuse notation and write:

$$n^2 \to \infty,$$

which reads "n^2 diverges to ∞". However, the definition of convergence (Definition 3.1) has not changed.

We write:

$$s_n \to \infty,$$

if for every $M\in\mathbb{R}$, there exists an integer N such that:

$$s_n \geq M, \quad \forall n \geq N$$

- Similarly, we write:
$$s_n \longrightarrow -\infty$$
if $\forall M \in \mathbb{R}$, $\exists N$ s.t:
$$s_n \leq M, \quad \forall n \geq N.$$

<u>Def</u>: Let $\{s_n\}$ be a sequence in $\mathbb{R}$. Define
$$E = \{x \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} : s_{n_k} \to x \text{ for some subsequence } \{s_{n_k}\}\}.$$

Note that E is the set of all subsequential limits of $\{s_n\}$ (denoted as E^*) plus possibly $+\infty$ or $-\infty$.

<u>Definition</u>: Let
$$s^* := \sup E$$
$$s_* := \inf E.$$

We write:
$$s^* = \limsup_{n \to \infty} s_n, \qquad s_* = \liminf_{n \to \infty} s_n$$

Clearly, $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$.

s^*, s_* are also called the upper and lower limits of s.

<u>Remark</u>: We have defined in previous lectures the <u>sup</u> (or <u>inf</u>) of a set bounded above (or (below). In this section we extend these concepts to unbounded sets, allowing s^* and s_* to take the values $\pm\infty$.

Ex:

$$s_n = \frac{(-1)^n}{1+\frac{1}{n}}$$

$$\frac{-1}{1+1},\ \frac{1}{1+\frac{1}{2}},\ \frac{-1}{1+\frac{1}{3}},\ \frac{1}{1+\frac{1}{4}},\ \frac{-1}{1+\frac{1}{5}},\ \cdots$$

We can extract two subsequences:

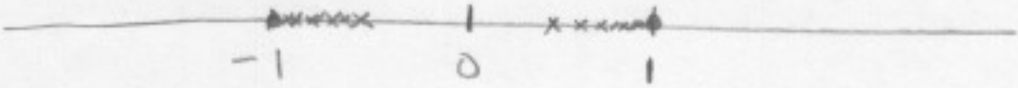
$\frac{-1}{1+1},\ \frac{-1}{1+\frac{1}{3}},\ \frac{-1}{1+\frac{1}{5}},\ \cdots$ which converges to -1.

$\frac{1}{1+\frac{1}{2}},\ \frac{1}{1+\frac{1}{4}},\ \frac{1}{1+\frac{1}{6}},\ \cdots,$ which converges to 1.

$\Rightarrow\ E^* = \{1, -1\},\quad E = E^* = \{1, -1\}$

$s^* = \sup E = 1 \quad , \quad s_* = \inf E$

$\Rightarrow\ \limsup_{n\to\infty} s_n = 1,\quad \liminf_{n\to\infty} s_n = -1$



Ex: Let $\{s_n\}$ be a sequence containing all rationals. Then:

$$\limsup_{n\to\infty} s_n = +\infty \quad , \quad \liminf_{n\to\infty} s_n = -\infty$$

Indeed, the subsequence $1, 2, 3, \cdots$ diverges to ∞

The subsequence $-1, -2, -3, \cdots,$ diverges to $-\infty$

Every rational number $\frac{p}{q}$ is the limit of the subsequence $\{\frac{p}{q}+\frac{1}{n}\}$ of $\{s_n\}$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, every irrational number is the limit of a subsequence of the sequence of rationals.

Hence $E = \{-\infty\} \cup \{\infty\} \cup \mathbb{R}$ and $s^* = \sup E = \infty,\ s_* = \inf E = -\infty$.

We have the following:

Theorem: (a) If $\{s_n\}$, $\{t_n\}$ are two sequences in $\mathbb{R}$ and $s_n \le t_n$, for all $n \geq N$, then:

$$\liminf_{n\to\infty} s_n \le \liminf_{n\to\infty} t_n$$

$$\limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} t_n$$

(b) If $\{s_n\}$ is a sequence in $\mathbb{R}$, then:

s_n coverges to $s \in R \iff \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = s$

In order to show the previous theorem, we need the following:

Theorem 3.17: With the same notation as in page 85:

(a) $s^* \in E$

(b) If $x > s^*$, there is an integer N such that $s_n < x$, for all $n \geq N$.

Moreover, s^* is the only number with the properties (a) and (b).

Remark: (b) says that all elements of $\{s_n\}$, except possibly for a finite number of them, are to the left of x.

Remark: A similar theorem holds for liminf.

Proof :

(a) :

Case 1 : If $s^* = +\infty$, then E is not bounded above and then it is clear that there exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ s.t. $s_{n_k} \to \infty$. Hence, $s^* \in E$.

Case 2 : If $s^* \neq \{+\infty, -\infty\}$ then $E = E^*$, where E^* is the set of all subsequential limits. In this case, E^* is bounded above. Theorem 2.28 implies that s^* belongs to the closure of E^*. In Theorem 3.7 we showed that E^* is closed. Hence $\overline{E^*} = E^*$ which gives $s^* \in E^*$. Since s^* is a subsequential limit, $\exists \{s_{n_k}\}_{k=1}^{\infty}$ such that $s_{n_k} \to s^*$. We conclude $s^* \in E$.

Case 3 : If $s^* = -\infty$, then E contains only one element, $-\infty$. In this case $E^* = \emptyset$; that is, there is no subsequential limits. This implies that for every $M \in \mathbb{R}$, there exists N such that:

$$s_n < M, \ \forall n \geq N. \qquad (1)$$

For otherwise, if this is not true, then $\exists M \in \mathbb{R}$ such that $\forall N$, $\exists n > N$ satisfying

$$s_n \geq M$$

Then, we would have a subsequence of $\{s_n\}$ that is bounded ($\{s_n\}$ must be bounded above, otherwise $s^* = \infty$) and the Bolzano-Weirstrass Theorem would imply the existence of a subsequential limit. Hence, (1) implies $s_n \to -\infty$, and $s^* \in E$.

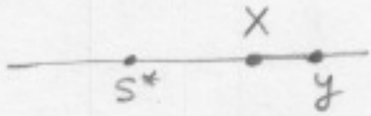
(b) If $s^* = \infty$, then $x = \infty$ and clearly $s_n < x$, $n = 1, 2, \dots$
If $s^* = -\infty$ then s_n diverges to $-\infty$ as we showed in <u>Case 3</u> of (a). Hence, for any $x > s^*$, there exists N such that $s_n < x$, $n \geq N$.

Assume now that $s^* \in \mathbb{R}$ (i.e., $-\infty < s^* < \infty$). We proceed by contradiction. Suppose that $\exists x > s^*$ such that:

$$s_n \geq x \quad \text{for infinitely many values of } n. \quad (1)$$

Since $s^* \in \mathbb{R}$, the sequence $\{s_n\}$ is bounded. Hence, from (1) and Bolzano-Weirstrass theorem there exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ that converges to a real number y, with:

$$y \geq x > s^* \quad (2)$$



But (2) contradicts that s^* is an upper bound for all subsequential limits (and y is a subsequential limit). We conclude that $\exists N$ s.t $s_n < x$, $n \geq N$.

<u>Uniqueness of limsup</u>: Assume that $\exists p, q$ with $p < q$ such that both satisfy (a) and (b). Choose x, $p < x < q$. From (b), $\exists N$ such that:

$$s_n < x, \quad \forall n \geq N. \quad (1)$$

But (1) implies that $q \notin E^*$, contradicting (a) ∎

Definition: A sequence $\{s_n\}$ in $\mathbb{R}$ is;

(a) monotonically increasing if;

$$s_n \leq s_{n+1} \quad (n = 1, 2, 3, \dots)$$

(b) monotonically decreasing if:

$$s_n \geq s_{n+1} \quad (n = 1, 2, 3, \dots)$$

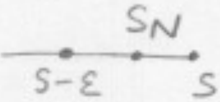
Theorem: Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Proof: Suppose $\{s_n\}$ is increasing:

$$s_n \leq s_{n+1}, \quad n = 1, 2, \dots$$

$\Leftarrow$ Let E be the range of $\{s_n\}$ and assume that E is bounded. Then $s := \sup E$ exists in $\mathbb{R}$. By definition of supremum:

$\forall \varepsilon > 0$, $\exists N$ such that $s - \varepsilon < s_N \leq s$,



for otherwise $s - \varepsilon$ would be a smaller upper bound of E.

Since $\{s_n\}$ increases, then:

$$s - \varepsilon < s_n \leq s, \quad \forall n \geq N$$

$$\Longrightarrow \quad s - s_n < \varepsilon, \quad \forall n \geq N$$

We have proved that $\forall \varepsilon > 0$, $\exists N$ s.t $|s - s_n| < \varepsilon$, $\forall n \geq N$,

Hence, $s_n \to s$.

$\Longrightarrow$ Assume now that $\{s_n\}$ converges, then Theorem 3.2 (c) shows it is bounded. ∎

Ex: Let $s_n = \frac{1}{\sqrt{n}}$.

Clearly, $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \Rightarrow \{s_n\}$ is decreasing. Since $\inf E = 0$ (E = range of $\{s_n\}$), we conclude $\frac{1}{\sqrt{n}} \to 0$.

Ex: Let $\{x_n\}$, $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, $n = 1, 2, \ldots$

Determine whether $\{x_n\}$ is convergent or divergent.

Solution: Note that:

$$x_{n+1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} = x_n + \frac{1}{n+1} > x_n$$

$\Rightarrow \{x_n\}$ is increasing

We have:

$$x_{2^n} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$

$$= 1 + \frac{n}{2}$$

Hence, $\{x_n\}$ is not bounded. The previous theorem implies that $\{x_n\}$ is divergent.