<u>Theorem 4.14</u>: If $f: X \to Y$ is continuous and X is a compact metric space then $f(X)$ is a compact subset of Y.

<u>Proof</u>: Let $\{G_\alpha\}$ be an upper cover of $f(X)$. Since f is continuous we have:

$$f^{-1}(G_\alpha) \text{ is open in } X, \ \forall \alpha$$

Note that:

$$X = \bigcup_\alpha f^{-1}(G_\alpha)$$

Since X is compact, there is a finite subcover:

$$\Rightarrow X \subset f^{-1}(G_{\alpha_1}) \cup \dots \cup f^{-1}(G_{\alpha_n}) \quad (1)$$

From (1) it follows that

$$f(X) \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}. \quad (2)$$

Indeed, in (2) we have used that $f(f^{-1}(E)) \subset E$, for any set E: if $x \in f(f^{-1}(E))$ then $x = f(z)$, for some $z \in f^{-1}(E)$, which implies that $f(z) \in E$. ∎

<u>Remark</u>: If $E \subset X$ is compact and $f: E \to Y$ is continuous, then $f(E)$ is compact.

<u>Note</u>: If E is any set, then:

$$E \subset f^{-1}(f(E)),$$

the proof is left to the reader.

<u>Theorem 4.15</u>: If $\vec{f}: X \to \mathbb{R}^k$, X compact, $\vec{f}$ continuos, then $\vec{f}(X)$ is closed and bounded. Thus, $\vec{f}$ is bounded.

<u>Proof</u>: From Theorem 4.14, $\vec{f}(X)$ is a compact set in $\mathbb{R}^k$. From Heine-Borel Theorem, $\vec{f}(X)$ is a closed and bounded set in $\mathbb{R}^k$. In particular, since $\vec{f}(X)$ is bounded. ∎

<u>Def</u>: $\vec{f}: E \subset X \to \mathbb{R}^k$ is said to be bounded if $\exists M$ such that $|\vec{f}(x)| \le M$, $\forall x \in E$.

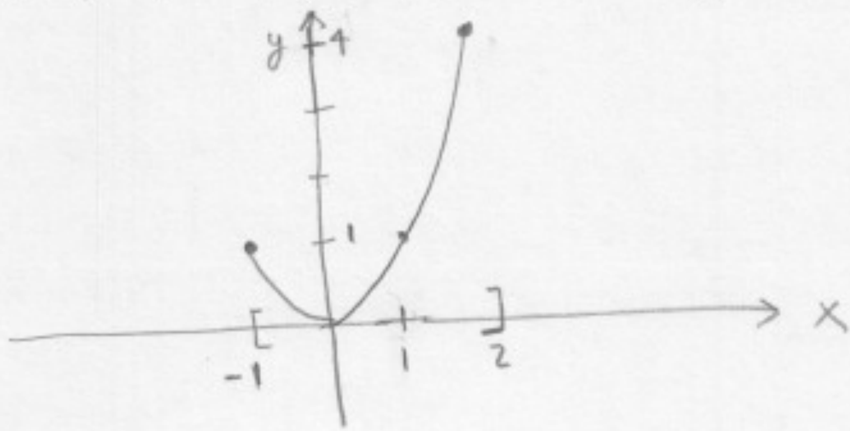
<u>Theorem 4.16</u>: Let $f: X \to \mathbb{R}$ be continuous, X compact, and:

$$M = \sup_{p \in X} f(p), \quad m = \inf_{p \in X} f(p)$$

Then, $\exists p, q \in X$ such that $f(p) = M$ and $f(q) = m$

<u>Remark</u>: Theorem 4.16 says that f attains its maximum and minimum at points in X.

<u>Ex</u>: $f(x) = x^2$, $f: [-1, 2] \to \mathbb{R}$ is continuous with $M = 4$, $m = 0$, $f(2) = 4$, $f(0) = 0$.



- Ex: Let $f: \mathbb{R} \to \mathbb{R}$
  $f(x) = e^x$

  $M = \infty$, $m = 0$, the sup and inf are not attained in $\mathbb{R}$

Proof of Theorem 4.16 : Since $f(X)$ is closed and bounded then, by Theorem 2.28, $\sup f(X)$ and $\inf f(X)$ belong to the closed set $f(X)$. Hence, $\exists\ p_1, p_2 \in X$ such that $M = f(p_1)$ and $m = f(p_2)$. ∎

Recall theorem 2.28 : E closed and bounded, $\alpha = \sup E$, then $\alpha \in \overline{E}$

Recall also that:

$$E \text{ closed} \iff \overline{E} = E$$

<u>Theorem 4.22</u> : Let $f: X \to Y$ be continuous. If E is a connected subset of X, then $f(E)$ is connected.

<u>Proof</u> : We proceed by contradiction. If $f(E)$ is not connected, then

$$\underline{f(E) = A \cup B}, \quad A, B \neq \emptyset, \quad A, B \subset Y$$
$$\underline{\bar{A} \cap B = \emptyset, \quad \bar{B} \cap A = \emptyset}$$

Define:
$$G = E \cap f^{-1}(A), \qquad H = E \cap f^{-1}(B)$$

We have:
$$\underline{E = G \cup H}, \quad G, H \neq \emptyset$$

Since $G \subset f^{-1}(A) \Rightarrow G \subset f^{-1}(\bar{A})$
$\Rightarrow \bar{G} \subset f^{-1}(\bar{A})$, because $f^{-1}(\bar{A})$ is closed
$\Rightarrow f(\bar{G}) \subset f(f^{-1}(\bar{A})) \subset \bar{A}$

Since $f(H) = B$ we have:
$f(\bar{G}) \cap f(H) \subset \bar{A} \cap B = \emptyset$.
Hence, $\underline{\bar{G} \cap H = \emptyset}$.

In the same way we prove that $\underline{G \cap \bar{H} = \emptyset}$

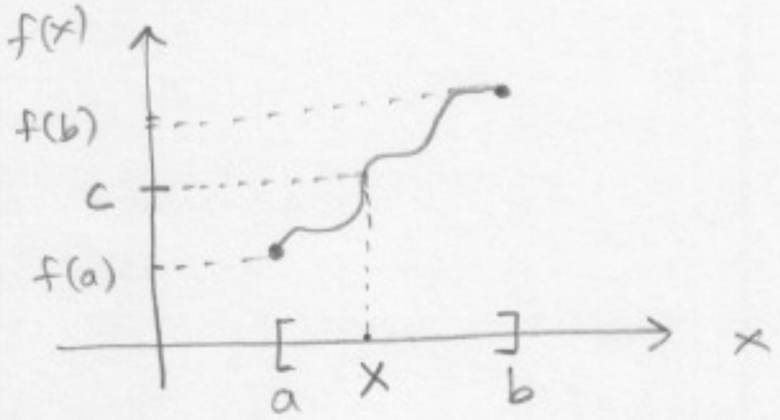
We have obtained the E is disconnected, which is a contradiction.

We conclude that E is connected. ∎

<u>Theorem 4.23</u>: Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a) < f(b)$ and $f(a) < c < f(b)$, then there exists $x \in (a,b)$ such that $f(x) = c$.

<u>Note</u>: This theorem says that a continuous real function assumes all intermediate values on an interval.

<u>Note</u>: A similar result holds if $f(a) > f(b)$.



<u>Proof</u>: Since $[a,b]$ is connected, Theorem 4.22 implies that $f([a,b])$ is connected in $\mathbb{R}^1$. In Theorem 2.47, we proved:

$E \subset \mathbb{R}^1$ is connected $\iff$ if $x, y \in E$, $x < z < y$, then $z \in E$.

We apply Theorem 2.47 with $x = f(a)$, $y = f(b)$, $z = c$ and $E = f([a,b]) = \{f(x) : a \le x \le b\}$.

Since $c \in E$, and $c \neq f(a), f(b)$, then there exists $a < x < b$ such that $f(x) = c$. ∎