<u>Def</u>: Let $f:(a,b)\to\mathbb{R}$

f is <u>monotonically increasing</u> on (a,b) if

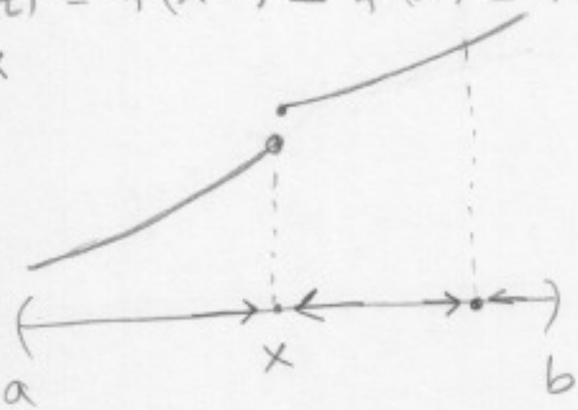
$$a<x<y<b \implies f(x)\le f(y)$$

f is <u>monotonically decreasing</u> on (a,b) if

$$a<x<y<b \implies f(x)\ge f(y)$$

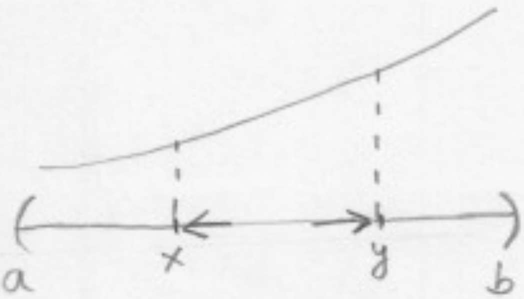
<u>Theorem 4.29</u>. Let $f:(a,b)\to\mathbb{R}$ be monotonically increasing on (a,b). Then $f(x+)$ and $f(x-)$ exist at every point $x\in(a,b)$. More precisely, we have that:

$$\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t)$$



Moreover, if $a<x<y<b$, then

$$f(x+)\le f(y-)$$



<u>Note</u>: A similar result holds for monotonically decreasing functions.

<u>Proof</u>: Let

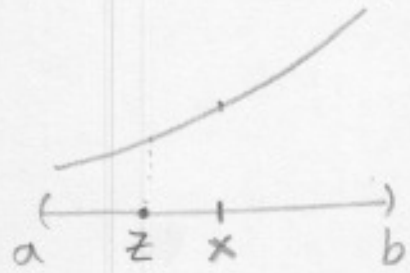
$$A = \sup_{a<t<x} f(t)$$

Since $f(t) \le f(x)$ $\forall t \in (a,x)$, then $f(x)$ is an upper bound for $\sup_{t\in(a,x)} f(t)$. Hence, since sup is the least upper bound, we obtain:

$$\boxed{A \le f(x)}$$

We proceed to show that $A = f(x-)$. Let $\varepsilon > 0$. By definition of A, there exists $y \in (a,x)$ such that:

$$A-\varepsilon < y \le A, \qquad y = f(z), \text{ for some } z \in (a,x)$$



Clearly, $z = x-\delta$, for some $\delta > 0$. Hence:

$$A-\varepsilon < f(x-\delta) \le A. \quad (1)$$

Since f is increasing:

$$f(x-\delta) \le f(t) \le A, \quad \forall t \in (x-\delta, x) \quad (2)$$

From (1) and (2) we obtain:

$$A-\varepsilon < f(x-\delta) \le f(t) \le A, \quad x-\delta < t < x,$$

that is:

$$|f(t) - A| < \varepsilon, \quad x-\delta < t < x$$

We have proved that:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t:

$$t \in (x-\delta, x) \Rightarrow |f(t) - A| < \varepsilon$$

Hence:

$$f(x-) = A.$$

We have proven:

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \qquad (3)$$

In the same way we obtain:

$$f(x) \leq f(x+) = \inf_{x<t<b} f(t) \qquad (4)$$

From (3) and (4) we conclude:

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$$

For the second part of the theorem we fix $x, y \in (a,b)$ such that:

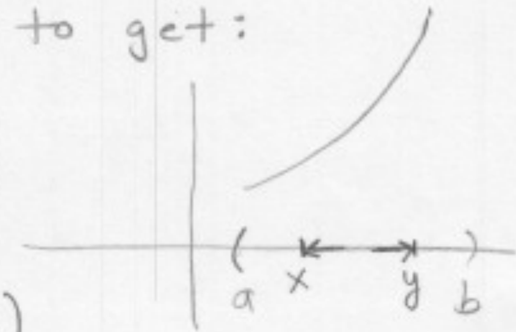
$$a < x < y < b$$

We apply the first part of the theorem to (a,y) instead of (a,b) to obtain:

$$f(x+) = \inf_{x<t<y} f(t) \qquad (5)$$

and to (x,b) instead of (a,b) to get:

$$f(y-) = \sup_{x<t<y} f(t) \qquad (6)$$



From (5) and (6), $f(x+) \leq f(y-)$

<u>Corollary</u>: Monotonic functions do not have discontinuities of the second kind (since $f(x+)$ and $f(x-)$ always exist)

<u>Theorem 4.30</u>: Let $f:(a,b)\to\mathbb{R}$ monotonic. Then the set of points of (a,b) at which f is discontinuous is at most countable.

<u>Proof</u>: Without loss of generality we consider the case f is increasing.

We define:

$$E = \{x \in (a,b) : f \text{ is discontinuous at } x\}$$

Let $x \in E$. Then, there exists a rational number $r(x)$ such that:

$$f(x-) < r(x) < f(x+)$$

Indeed, from previous theorem, $f(x-) \le f(x) \le f(x+)$. If $f(x-) = f(x+) = f(x)$, then f is continuous at x. Hence, since $x \in E$, we must have $f(x-) < f(x+)$, and therefore, the interval $(f(x-), f(x+))$ contains a rational number.

If $x_1 \neq x_2$, say $x_1 < x_2$:

$$f(x_1-) < r(x_1) < f(x_1+) \le f(x_2-) < r(x_2) < f(x_2+),$$

and hence $r(x_1) < r(x_2) \Rightarrow r(x_1) \neq r(x_2)$.

Hence, E is equivalent to a subset of the rational number $\mathbb{Q}$. That is, $\exists\ f: E \to S$ 1-1 and on-to, $S \subset \mathbb{Q}$. Since $\mathbb{Q}$ is countable, S is at most countable. $\blacksquare$

Infinite limits and limits at infinity

We recall definition 4.1: X, Y metric spaces, $f: E \subset X \to Y$, $p \in E'$. Then,

$$\lim_{x\to p} f(x) = q,\ q \in Y \text{ if } \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.:}$$

$$x \in E,\ d_X(x,p) < \delta \implies d_Y(f(x), q) < \varepsilon.$$

For the case $f: E \subset \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{x\to p} f(x) = \ell \quad \text{if } \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ such that:}$$

$$\text{if } |x - p| < \delta,\ x \in E \implies |f(x) - \ell| < \varepsilon. \quad (1$$

In (1), both p and ℓ belong to $\mathbb{R}$. Sometime we need take limits when $\ell \in \{-\infty, +\infty\}$ and/o $p \in \{+\infty, -\infty\}$. We consider the extended real system $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. For example,

$$f(x) \longrightarrow \infty \quad \text{as} \quad x \longrightarrow a, \quad a \in \mathbb{R}$$

means:

$\forall M \in \mathbb{R}, \exists \delta > 0$ s.t.

if $|x - a| < \delta$, $x \in E$ then $f(x) > M$.

$f(x) \to a$ as $x \to \infty$, $a \in \mathbb{R}$

means:

$\forall \varepsilon > 0$, $\exists M \in \mathbb{R}$ such that:

For all $x \in E$, $x > M$ we have $|f(x) - a| < \varepsilon$.

$f(x) \to \infty$ as $x \to \infty$ means:

$\forall M \in \mathbb{R}$, $\exists N \in \mathbb{R}$ such that:

For all $x \in E$, $x > M$ we have $f(x) > N$.

We have similar definitions in all other situations involving $\pm\infty$.

We remark that, with these types of limits, we have:

Theorem 4.34. Let $f: E \subset \mathbb{R} \to \mathbb{R}$, $g: E \subset \mathbb{R} \to \mathbb{R}$.
Suppose $f(t) \to A$, $g(t) \to B$ as $t \to x$,
where x, A, B belong to $\{-\infty\} \cup \{+\infty\} \cup \mathbb{R}$.
Then, as $t \to x$, we have:
(a) $f(t) \to A'$ implies $A' = A$
(b) $(f+g)(t) \to A+B$
(c) $(fg)(t) \to AB$
(d) $(f/g)(t) \to A/B$,
provided the right members of (b), (c), and (d) are defined. ($\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{A}{0}$ are not defined)

Ex.

$$f(x) = \frac{x^2}{1+x^2}$$

We have $f(x) = \dfrac{1}{\frac{1}{x^2}+1}$.

$\frac{1}{x^2} \to 0$ as $x \to \infty$ because $\forall \varepsilon > 0, \exists N$ s.t:

if $x > N \Rightarrow \left| \frac{1}{x^2} - 0 \right| < \varepsilon$.

From theorem 4.34 (b), $1 + \frac{1}{x^2} \to 1 + 0 = 1$ as $x \to \infty$.

From theorem 4.34 (d), $\dfrac{1}{\frac{1}{x^2}+1} \to \frac{1}{1} = 1$ as $x \to \infty$.

<u>Thm</u>: Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping $f^{-1}: Y \to X$ given by:

$$f^{-1}(f(x)) = x, \quad x \in X$$

is a continuous mapping.

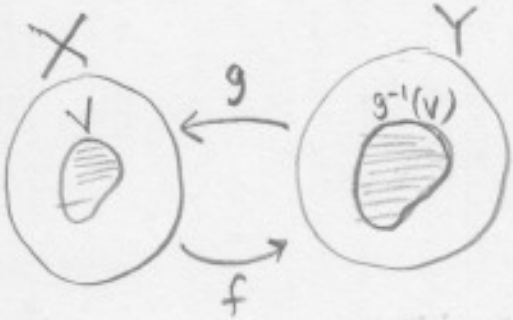
<u>Proof</u>: We have:

$f: X \to Y$, f is injective and surjective.

$f^{-1}: Y \to X$

Let V open in X, let $g = f^{-1}$. We need to show that $g^{-1}(V)$ is open in Y. But:

$$g^{-1}(V) = \{ y \in Y : g(y) \in V \}.$$



Since $f(V) = g^{-1}(V)$, it is enough to show that $f(V)$ is open.

Since V^c is closed, and $V^c \subset X$, X compact, Theorem 2.35 yields that V^c is compact.

Since f is continuous, $f(V^c)$ is compact. Hence, $f(V^c)$ is closed. But $f(V) = (f(V^c))^c$, since f is 1-1 and onto. This implies that $f(V)$ is open. ∎

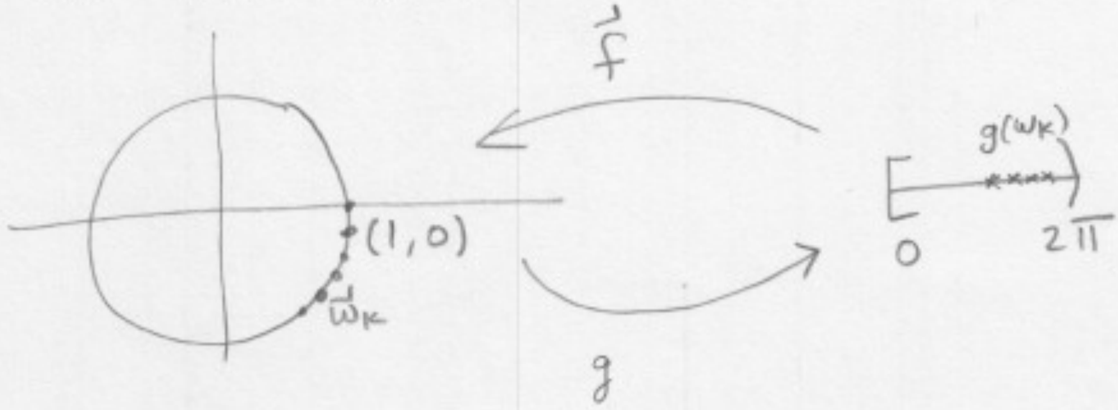
<u>Ex</u> ; $X = [0, 2\pi)$

$$\vec{f} : X \to \mathbb{R}^2$$

$$f(t) = (\cos t, \sin t)$$

$\vec{f}$ is continuous since $t \mapsto \sin t$, $t \mapsto \cos t$ are continuous (this is proven in chapter 8, but we can use it).

$\vec{f}$ is a continuous 1-1 map of $[0, 2\pi)$ onto the unit circle.



Let $g = (\vec{f})^{-1}$.

Note that X is not compact.

We have that g is not continuous at $(1,0)$ since we can construct a sequence $\vec{\omega}_k \to (1,0)$ such that:

$$\lim_{k\to\infty} g(\vec{\omega}_k) = 2\pi \neq \vec{g}(1,0) = 0. \quad \blacksquare$$

<u>Note</u>: Recall that if $f: E \subset X \to Y$ is continuous, and $\{x_n\}$ is a sequence in E and $x_n \to x$, $x \in E$, then:

$$f(x_n) \to f(x)$$

Indeed, let $\varepsilon > 0$.

Since f is continuos at x, there exists $\delta > 0$ such that:

$$d_X(y, x) < \delta,\ y \in E \implies d_Y(f(y), f(x)) < \varepsilon \quad (1)$$

Now, since $x_n \to x$, $\exists N$ such that:

$$d_X(x_n, x) < \delta, \quad \forall n \geq N \quad (2)$$

From (1) and (2) it follows that:

$$d_Y(f(x_n), f(x)) < \varepsilon, \quad \forall n \geq N.$$

We have shown that, given any $\varepsilon > 0$, $\exists N$ such that $d_Y(f(x_n), f(x)) < \varepsilon$, $\forall n \geq N$. This means:

$$f(x_n) \to f(x) \quad \longrightarrow (**)$$

<u>Ex</u>: Use $(**)$ and the convergence $\frac{\ln n}{n} \to 0$ as $n \to \infty$, to show that $\sqrt[n]{n} \to 1$ as $n \to \infty$.

<u>Solution</u>: Let $y_n = \ln \sqrt[n]{n} = \frac{1}{n} \ln n = \frac{\ln n}{n}$.

Hence, $\lim_{n \to \infty} y_n = 0$. Consider the continuous function $f(x) = e^x$. Since $y_n \to 0$ then $f(y_n) \to f(0)$. That is,

$$e^{y_n} \to e^0$$

Therefore, $e^{\ln \sqrt[n]{n}} \to 1 \implies \sqrt[n]{n} \to 1.$

<u>Note</u>: In chapter 5, we will prove L'Hospital's Rule, from which we obtain easily that $\frac{\ln n}{n} \to 0$ as $n \to \infty$.