Chapter 5
Differentiation

In this chapter we will work with with real valued functions $f: \mathbb{R} \to \mathbb{R}$.

Def: Let $f:[a,b] \to \mathbb{R}$. Let $x \in [a,b]$. We form:

$$\phi(t) = \frac{f(t)-f(x)}{t-x}, \quad a<t<b, \; t \neq x.$$

We define:

$$f'(x) = \lim_{t \to x} \phi(t),$$

provided that this limit exists (as in Definition 4.1).

Remark: If f' is defined at a point x, we say that f is differentiable at x. If f' is defined at every point of a set $E \subset [a,b]$, we say that f is differentiable on E. At the endpoints a and b, if the derivative exists, it is a right-hand or left-hand derivative, respectively.

Theorem 5.2: Let $f: [a,b] \to \mathbb{R}$. If f is differentiable at $x \in [a,b]$ then f is continuous at x.

Proof: We have:

$$f(t)-f(x) = \frac{f(t)-f(x)}{t-x} \cdot (t-x)$$

$$\lim_{t \to x} (f(t)-f(x)) = \lim_{t \to x} \left[\frac{f(t)-f(x)}{t-x} \cdot (t-x)\right] = f'(t) \cdot 0 = 0,$$

from Theorem 4.4. Hence, $\lim_{t \to x} f(t) = f(x)$. ∎

Theorem 5.3. Let $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$. If both f and g are differentiable at $x\in[a,b]$, then $f+g$, fg, $\frac{f}{g}$ are also differentiable at x, with:

(a) $(f+g)'(x) = f'(x) + g'(x)$

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

(c) $\left(\frac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}$, $g(x)\neq 0$.

Proof: We will prove (b). The others are left to the reader.

Let $h = fg$. Then:

$$h(t) - h(x) = f(t)\,[g(t) - g(x)] + g(x)\,[f(t) - f(x)]$$

$$\Rightarrow \quad \frac{h(t)-h(x)}{t-x} = f(t)\left[\frac{g(t)-g(x)}{t-x}\right] + g(x)\left[\frac{f(t)-f(x)}{t-x}\right]$$

$$\lim_{t\to x}\frac{h(t)-h(x)}{t-x} = \left[\lim_{t\to x} f(t)\right]\left[\lim_{t\to x}\frac{g(t)-g(x)}{t-x}\right] + g(x)\lim_{t\to x}\left[\frac{f(t)-f(x)}{t-x}\right];$$ From Theorem 4.4

Since f, g are differentiable at x, using Theorem 5.2, we conclude:

$$\lim_{t\to x}\frac{h(t)-h(x)}{t-x} = f(x)\cdot g'(x) + g(x)f'(x)$$

$$\Rightarrow \quad h'(x) = f(x)g'(x) + g(x)f'(x).$$

Ex: $f(x)=x$ is differentiable since

$$\lim_{t\to x}\frac{f(t)-f(x)}{t-x}=\lim_{t\to x}\frac{t-x}{t-x}=1.$$

From Theorem 5.3, $f(x)=x^2=x\cdot x$ is also differentiable, and $f'(x)=x\cdot 1+x\cdot 1=2x$. By repeated application of theorem 5.3, it follows that $f(x)=x^n$ is differentiable and $f'(x)=nx^{n-1}$, and that every polynomial is differentiable. Then, by Theorem 5.3 (c), every rational function (i.e., a quotient of polynomials) is also differentiable, except at the points where the denominator is zero.

Theorem 5.5 (Chain rule): Let $f:[a,b]\to\mathbb{R}$ continuous, suppose $f'(x)$ exists at some point $x\in[a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If:

$$h(t)=g(f(t)),\quad a\le t\le b,$$

then h is differentiable at x, and

$$h'(x)=g'(f(x))\,f'(x).$$

Ex: Let:

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

It is proven in Chapter 8 that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$. For $x \neq 0$, we can use Theorem 5.3 and 5.5 to obtain:

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}, \quad x \neq 0.$$

Hence, f is differentiable at any $x \neq 0$. If $x=0$, we use the definition:

$$\lim_{t \to 0} \frac{f(t)-f(0)}{t-0} = \lim_{t \to 0} \frac{t \sin \frac{1}{t}}{t} = \lim_{t \to 0} \sin \frac{1}{t},$$

which does not exist. Hence f is not differentiable at 0.

Ex Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

If $x \neq 0$, Theorem 5.3 and 5.5 yield:

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

So f is differentiable at $x \neq 0$. If $x=0$, we need to use the definition of differentiability.

$$0 \leq \left| \frac{f(t)-f(0)}{t-0} \right| = \left| t \sin \frac{1}{t} \right| \leq |t|, \quad t \neq 0$$

The squeeze lemma implies that:

$$\lim_{t \to 0} \left| \frac{f(t)-f(0)}{t-0} \right| = 0;$$

and hence $\lim_{t \to 0} \frac{f(t)-f(0)}{t-0} = 0$. We conclude

that $f'(0) = 0$. Hence, f is differentiable
at all points x. Notice that f' is not continuous at $x=0$ since:

$$\lim_{x\to 0} f'(x) = \lim_{x \to 0} \left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right) \text{ does}$$

not exist. Indeed, $\lim_{x\to 0} 2x \sin\frac{1}{x} = 0$, but $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist.

Mean value theorems

Definition 5.7: Let $f: X \to \mathbb{R}$. We say that f has a local maximum at $p \in X$ if $\exists\, \delta > 0$ such that:

$$f(q) \le f(p), \quad \forall q \in X,\ d(q,p) < \delta$$

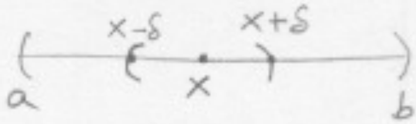
f has a local minimum at $p \in X$ if $\exists\, \delta > 0$ such that:

$$f(q) \geq f(p), \quad \forall q \in X,\ d(q,p) < \delta.$$

Theorem 5.8: Let $f: [a,b] \to \mathbb{R}$. If f has a local maximum at $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$. (The analogous statement for local minima is also true).

Proof: Let $\delta > 0$ be given by Definition 5.7.

$$\Rightarrow \quad a < x - \delta < x < x + \delta < b$$



If $t \in (x-\delta, x) \Rightarrow \dfrac{f(t) - f(x)}{t - x} \geq 0$; since $f(t) \le f(x)$

Thus, $\lim_{t \to x} \dfrac{f(t) - f(x)}{t - x} \geq 0 \Rightarrow f'(x) \geq 0$.

If $t \in (x, x+\delta) \Rightarrow \dfrac{f(t) - f(x)}{t-x} \le 0 \Rightarrow f'(x) \le 0$

Since $0 \le f'(x) \le 0$ we conclude $f'(x) = 0$. ∎

Theorem 5.9 : Let $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ continuous functions which are differentiable in (a,b), then there is a point $x\in(a,b)$ at which:

$$[f(b)-f(a)]g'(x)=[g(b)-g(a)]f'(x)$$

Note that differentiability is not required at the end points.

Proof: Define:

$$h(t)=[f(b)-f(a)]g(t)-[g(b)-g(a)]f(t),\quad a\le t\le b$$

h is continuous on $[a,b]$

h is differentiable in (a,b).

$$h(a)=f(b)g(a)-\cancel{f(a)g(a)}-g(b)f(a)+\cancel{g(a)f(a)}$$
$$=f(b)g(a)-g(b)f(a)$$

$$h(b)=\cancel{f(b)g(b)}-f(a)g(b)-\cancel{g(b)f(b)}+g(a)f(b)$$
$$=f(b)g(a)-g(b)f(a)$$

$\Rightarrow h(a)=h(b)$.

If $\exists x\in(a,b)$ such that $h'(x)=0$, then

$$0=h'(x)=[f(b)-f(a)]g'(x)-[g(b)-g(a)]f'(x)$$

and hence we obtain the desired result:

$$[f(b)-f(a)]g'(x)=[g(b)-g(a)]f'(x)\quad(*)$$

In order to show the existence of such x we consider:

Case 1 : If $h(t)$ is a constant function, then $h'(x)=0$, for every $x\in(a,b)$, and $(*)$ holds for every $x\in(a,b)$.

Case 2: If h is not constant and $h(t)>h(a)$ for some $t\in(a,b)$ then, since $[a,b]$ is compact, h attains its maximum at some point $x\in[a,b]$. Clearly, $x\neq a,b$. Since h has a local maximum at x, from Theorem 5.8 we obtain that $f'(x)=0$

Case 3: If h is not constant and $h(t)<h(a)$ for some $t\in(a,b)$ then, since $[a,b]$ is compact, h attains its minimum at some point $x\in[a,b]$. Clearly, $x\neq a,b$. Since h has a local minimum at x, from Theorem 5.8 we obtain that $f'(x)=0$

In any case, we found $x\in(a,b)$ with $f'(x)=0$. The desired result holds at this x.

Theorem 5.10 (Mean value Theorem): Let $f:[a,b]\to\mathbb{R}$ a continuous function which is differentiable in (a,b), then $\exists\, x\in(a,b)$ such that

$$f(b)-f(a)=f'(x)\,(b-a)$$

Proof: Let $g(x)=x$

Applying previous theorem we get that $\exists\, x\in(a,b)$ satisfying:

$$[f(b)-f(a)]\,g'(x)=[g(b)-g(a)]\,f'(x)$$

$$\Rightarrow [f(b)-f(a)]\cdot 1=(b-a)\,f'(x)$$; since $g'(x)=1$, $g(b)=b$, and $g(a)=a$.

We conclude:

$$f(b)-f(a)=f'(x)\,(b-a)$$

Theorem 5.11 : Let $f : (a,b) \to \mathbb{R}$, f differentiable in (a,b). We have:

(a) If $f'(x) \geq 0$ $\forall x \in (a,b)$ then f is monotonically increasing

(b) If $f'(x) = 0$ $\forall x \in (a,b)$ then f is constant.

(c) If $f'(x) \leq 0$ $\forall x \in (a,b)$ then f is monotonically decreasing.

Proof : Let $x_1, x_2 \in (a,b)$, $x_1 < x_2$.

From the mean value theorem, $\exists x \in (x_1, x_2)$ such that:

$$f(x_2) - f(x_1) = f'(x)\,(x_2 - x_1) \qquad (*)$$

(a) Since $f'(x) \geq 0$ and $x_2 - x_1 > 0$, then from $(*)$:

$f(x_2) - f(x_1) \geq 0 \Rightarrow f(x_1) \leq f(x_2) \Rightarrow f$ is increasing.

(c) Since $f'(x) \leq 0$ and $x_2 - x_1 > 0$, then from $(*)$:

$f(x_2) - f(x_1) \leq 0 \Rightarrow f(x_1) \geq f(x_2) \Rightarrow f$ is decreasing.

(b) Since $f'(x) = 0$, then from $(*)$:

$f(x_2) = f(x_1) \Rightarrow f$ is constant.

We have proven the theorem since x_1, x_2 where arbitrary points in (a,b).