- Theorem 5.13 (L'Hospital rule):

Let f, g be real valued functions

- Suppose that:

$$\frac{f'(x)}{g'(x)} \to A \quad \text{as} \quad x \to a \quad (1)$$

If

$$f(x) \to 0 \text{ and } g(x) \to 0 \quad \text{as} \quad x \to a$$

or if:

$$g(x) \to \infty \quad \text{as } x \to a$$

then:

$$\frac{f(x)}{g(x)} \to A \quad \text{as} \quad x \to a.$$

The theorem is valid for limits in the extended real system. That is:

$$A \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}, \quad a \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

Proof:

Case 1: $A \in \mathbb{R}$, $a \in \mathbb{R}$

Let $\varepsilon > 0$.

Since $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = A$, $\exists \delta$ s.t

$$|x-a| < \delta, \; x \neq a \implies \left| \frac{f'(x)}{g'(x)} - A \right| < \frac{\varepsilon}{2}$$

that is:

$$x \in (a-\delta, a) \cup (a, a+\delta) \Rightarrow A - \frac{\varepsilon}{2} < \frac{f'(x)}{g'(x)} < A + \frac{\varepsilon}{2}, \quad (2)$$

Suppose now that:

$$\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0,$$

For $a < s < x < a+\delta$, the generalized mean value theorem implies that $\exists\, t \in (s, x)$ such that:

$$\frac{f(x) - f(s)}{g(x) - g(s)} = \frac{f'(t)}{g'(t)} < A + \frac{\varepsilon}{2}, \quad \text{by } (2)$$

$$\therefore \quad \frac{f(x) - f(s)}{g(x) - g(s)} < A + \frac{\varepsilon}{2}, \ \forall s, x, \quad a < s < x < a+\delta$$

We fix x, and let $s \to a$:

$$\lim_{s \to a} \frac{f(x) - f(s)}{g(x) - g(s)} = \frac{f(x) - \lim_{s \to a} f(s)}{g(x) - \lim_{s \to a} g(s)} = \frac{f(x)}{g(x)} \leq A + \frac{\varepsilon}{2}$$

Since x is arbitrary:

$$\boxed{\frac{f(x)}{g(x)} < A + \varepsilon, \quad \forall x, \quad a < x < a+\delta} \quad (A)$$

Similarly, for $a - \delta < x < z < a$, $\exists\, t \in (x, z)$ s.t:

$$\frac{f(z) - f(x)}{g(z) - g(x)} = \frac{f'(t)}{g'(x)} > A - \frac{\varepsilon}{2}, \quad \text{by } (2)$$

$$\lim_{z \to a} \frac{f(z) - f(x)}{g(z) - g(x)} = \frac{\lim_{z \to a} f(z) - f(x)}{\lim_{z \to a} f(z) - g(x)} = \frac{f(x)}{g(x)} \geq A - \frac{\varepsilon}{2} > A - \varepsilon, \text{ and}$$

Since x is arbitrary:

$$\boxed{\frac{f(x)}{g(x)} > A - \varepsilon, \quad \forall x, \quad a - \delta < x < a} \quad (B)$$

Similarly, for $a < s < x < a+\delta$, $\exists t \in (s,x)$ s.t:

$$\frac{f(x)-f(s)}{g(x)-g(s)} = \frac{f'(t)}{g'(t)} > A - \frac{\varepsilon}{2}, \text{ by } (2)$$

$$\lim_{s \to a} \frac{f(x)-f(s)}{g(x)-g(s)} = \frac{f(x)}{g(x)} \geq A - \frac{\varepsilon}{2} > A - \varepsilon$$

$$\Rightarrow \boxed{\frac{f(x)}{g(x)} > A-\varepsilon, \quad \forall x, \quad a < x < a+\delta} \quad (A')$$

Similarly, for $a-\delta < x < z < a$, $\exists t \in (x,z)$ s.t:

$$\frac{f(z)-f(x)}{g(z)-g(x)} = \frac{f'(t)}{g'(t)} < A + \frac{\varepsilon}{2} \; ; \text{ by } (2)$$

$$\lim_{z \to a} \frac{f(z)-f(x)}{g(z)-g(x)} = \frac{f(x)}{g(x)} \leq A + \frac{\varepsilon}{2} < A + \varepsilon$$

$$\Rightarrow \boxed{\frac{f(x)}{g(x)} < A+\varepsilon, \quad \forall x, \quad a-\delta < x < a} \quad (B')$$

From (A), (A'), (B), (B') we conclude

$$A-\varepsilon < \frac{f(x)}{g(x)} < A+\varepsilon \quad \text{if } |x-a| < \delta, \; x \neq a$$

That is:

$$\left| \frac{f(x)}{g(x)} - A \right| < \varepsilon, \quad \text{if } |x-a| < \delta, \; x \neq a,$$

which says:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = A.}$$

We assume now that:

$$\lim_{x\to\infty} g(x) = \infty$$

Using the generalized mean value theorem as above, for $a < s < x < a+\delta$, we have:

$$\frac{f(x)-f(s)}{g(x)-g(s)} < A + \frac{\varepsilon}{2} \quad (*)$$

Fix s. Since $g(x) \to \infty$ as $x \to a$, for x close enough to a, $\frac{g(x)-g(s)}{g(x)} > 0$. We multiply both sides of $(*)$ by $\frac{g(x)-g(s)}{g(x)}$:

$$\frac{\cancel{g(x)-g(s)}}{g(x)} \cdot \frac{f(x)-f(s)}{\cancel{g(x)-g(s)}} < \left(A+\frac{\varepsilon}{2}\right) \cdot \frac{g(x)-g(s)}{g(x)}$$

$$\Rightarrow \quad \frac{f(x)}{g(x)} - \frac{f(s)}{g(x)} < \left(A+\frac{\varepsilon}{2}\right)\left(1-\frac{g(s)}{g(x)}\right)$$

$$\Rightarrow \quad \frac{f(x)}{g(x)} < \left(A+\frac{\varepsilon}{2}\right)\left(1-\frac{g(s)}{g(x)}\right) + \frac{f(s)}{g(x)}$$

Since $g(x) \to \infty$ as $x \to a$ we have:

$$\boxed{\begin{array}{l} \forall M>0,\ \exists \delta_1 \text{ s.t.}: \qquad (3) \\ |x-a| < \delta_1,\ x \neq a \Rightarrow g(x) > M \end{array}}$$

it follows that $\exists \delta_1$ such that $\delta_1 < \delta$ and:

$$\left(A+\frac{\varepsilon}{2}\right)\left(1-\frac{g(s)}{g(x)}\right)+\frac{f(s)}{g(x)}<A+\varepsilon,$$

for $x \in (a, a+\delta_1)$.

Hence:

$$\boxed{\frac{f(x)}{g(x)}<A+\varepsilon,\ x\in(a,a+\delta_1)} \quad (C).$$

Using the same arguments with three more cases, we obtain that $\exists\, \tilde{\delta} < \delta_1$ such that

$$\frac{f(x)}{g(x)} < A+\varepsilon, \quad x \in (a-\tilde{\delta}, a) \quad (C')$$

$$\frac{f(x)}{g(x)} > A-\varepsilon, \quad x \in (a, a+\tilde{\delta}) \quad (D)$$

$$\frac{f(x)}{g(x)} > A-\varepsilon, \quad x \in (a-\tilde{\delta}, a) \quad (D')$$

From (C), (C'), (D), (D') we conclude:

$$|x-a|<\tilde{\delta},\ x\neq a \implies A-\varepsilon < \frac{f(x)}{g(x)} < A+\varepsilon,$$

$$\implies \left|\frac{f(x)}{g(x)}-A\right|<\varepsilon.$$

Therefore:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = A.$$

All the other cases are proven with similar arguments, using the definitions of limits in the extended real system (see page 132 in Lecture notes).

Case 2 : $A = -\infty$ or $A = +\infty$; $a \in \mathbb{R}$

Case 3 : $A \in \mathbb{R}$, $a = -\infty$ or $a = +\infty$

Case 4 : $A \in \{-\infty, +\infty\}$ and $a \in \{-\infty, +\infty\}$

Ex : Compute $\lim_{x \to 0} \frac{\sin x}{x}$.

Let $f(x) = \sin x$, $g(x) = x$

$\sin x \to 0$ as $x \to 0$, $g(x) \to 0$ as $x \to 0$

Since $\frac{f'(x)}{g'(x)} = \frac{\cos x}{1} \to 1$ as $x \to 0$, previous theorem yields:

$$\frac{\sin x}{x} \to 1 \quad \text{as } x \to 0.$$

Ex : Compute $\lim_{x \to \infty} \frac{\ln x}{x}$

Let $f(x) = \ln x$, $g(x) = x$

$f(x) \to \infty$ as $x \to \infty$, $g(x) \to \infty$ as $x \to \infty$

Since $\frac{f'(x)}{g'(x)} = \frac{1}{x} \to 0$ as $x \to \infty$, previous theorem gives $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$.

Derivatives of higher order

Theorem 5.15 : Let $f : [a,b] \to \mathbb{R}$, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t \in (a,b)$.

Let $\alpha, \beta \in [a,b]$. Define:

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t-\alpha)^k .$$

Then, $\exists\, x$ between α and β such that:

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n .$$

This theorem says that f can be approximated by a polynomial of degree $n-1$, and that the error is:

$$\frac{f^{(n)}(x)}{n!} (\beta-\alpha)^n ,$$

which can be estimated if we have bounds for $|f^{(n)}(x)|$.

Proof: If $n=1$, this theorem is just the mean value theorem. Indeed:

$$P(t) = \frac{f^{(0)}(\alpha)}{0!}(t-\alpha)^0 = f(\alpha) \Rightarrow P(\beta) = f(\alpha)$$

$f(\beta) = f(\alpha) + f'(x)(\beta - \alpha)$ is the mean value theorem.

- Let M be the number defined by:

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n$$

Define:

$$g(t) = f(t) - P(t) - M(t-\alpha)^n, \quad a \le t \le b$$

We need to show that:

$$M = \frac{f^{(n)}(x)}{n!}, \text{ for some } x \text{ between } \alpha \text{ and } \beta.$$

Note:

$$(*) \ \boxed{g^{(n)}(t) = f^{(n)}(t) - n!\,M}, \quad a < t < b.$$

Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$, $k = 0, 1, \dots, n-1$,

we have:

$$g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0.$$

Our choice of M shows that $g(\beta) = 0$.

$\Rightarrow \quad g'(x_1) = 0$ for some x_1 between α and β, by the mean value theorem.

$$g'(\alpha) = 0 \Rightarrow g''(x_2) = 0 \quad \text{for some } x_2$$

between α and x_1, by the mean value theorem.

After n steps we conclude $g^{(n)}(x_n) = 0$, for some x_n between α and x_{n-1}, that is, between α and β. From $(*)$:

$$0 = g^{(n)}(x_n) = f^{(n)}(x_n) - n!\,M \Rightarrow M = \frac{f^{(n)}(x_n)}{n!}. \ \blacksquare$$