Real numbers

The real numbers are represented by $\mathbb{R}$.
$\mathbb{R}$ is the union of two sets: the rational numbers and the irrational numbers. That is:

$$\mathbb{R} = \{\text{set of rational numbers}\} \cup \{\text{set of irrational numbers}\}$$

The set of rational numbers is represented by $\mathbb{Q}$ and are those numbers of the form $\frac{m}{n}$, where m and n are integers, $n \neq 0$.

Examples of rational numbers:

$\frac{1}{2} = 0.5$ $\qquad$ $\frac{10}{3} = 3.3333\ldots$

$\frac{7}{8} = 0.875$ $\qquad$ $\frac{1}{7} = 0.142857142857\ldots$

$\frac{1}{6} = 0.1666\ldots$ $\qquad$ $3, -5, 0$ (all the integers are also rational numbers).

Remark: Any number with finite decimal representation is a rational number.

Ex: $0.875 = \frac{875}{1000} = \frac{7}{8}$.

The rational number system has gaps. That is, not every real number can be written as the quotient of two integers. The irrational numbers fill these gaps and are constructed as the limit of rational numbers.

Ex: 1, 1.4, 1.41, 1.414, 1.4142, ...
tends to $\sqrt{2}$, which is an irrational number.

The next example shows that $\sqrt{2} = 1.414213562\ldots$ is irrational.

Ex: Show that the equation $p^2 = 2$ is not satisfied for any rational p.

Proof: We proceed by contradiction and assume that there exists a rational number $p = \frac{m}{n}$ (where m, n are not both even) such that:

$$p^2 = 2$$

Hence, $\frac{m^2}{n^2} = 2$. That is, $m^2 = 2n^2$.

$$\Rightarrow \quad \frac{m \cdot m}{2} = n^2 \qquad (2)$$

Since $\frac{m \cdot m}{2}$ must be an integer, it follows that m must be even. Hence;

$$m = 2K, \text{ for some integer } K. \quad (3)$$

We substitute (3) back in (2) to obtain:

$$\frac{(2K)(2K)}{2} = n^2$$

That is, $2K^2 = n^2$. Again, from:

$$K^2 = \frac{n \cdot n}{2},$$

it follows that n must be even too, which is a contradiction, since $p = \frac{m}{n}$ was simplified, if necessary, at the beginning of the proof so that m, n are not both even. ∎

Definitions:

If A is any set (of numbers or any other objects), we write $x \in A$ to indicate that x is an element of A. If x is not a member of A, we write $x \notin A$.

If a set does not have any elements, it is called <u>empty set</u>.

If a set has at least one element, it is called <u>non-empty</u>.

If A and B are sets, and if every element of A is also an element of B, we say that A is a subset of B, and we write:

$$A \subset B, \quad \text{or} \quad B \supset A.$$

If, in addition, there is an element of B which is not in A, then we say:

A is a <u>proper subset</u> of B.

Note that $A \subset A$ for every set A.

<u>Remark</u>: $A = B$ if and only if $A \subset B$ and $B \subset A$.

Otherwise, $A \neq B$.

Some properties of real numbers

Theorem 1: Let a, b, c, d be elements of $\mathbb{R}$. Then:

(a) If $a > b$, then $a + c > b + c$

(b) If $a > b$ and $c > d$, then $a + c > b + d$

(c) If $a > b$ and $c > 0$, then $ac > bc$

(c') If $a > b$ and $c < 0$, then $ac < bc$

(d) If $a > 0$, then $\frac{1}{a} > 0$

(d') If $a < 0$, then $\frac{1}{a} < 0$

Def: If $a \in \mathbb{R}$, the absolute value of a is denoted by $|a|$ and is defined by:

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

The following is true:

Theorem 2:

(a) $|a| = 0$ if and only if $a = 0$.

(b) $|-a| = |a|$, for all $a \in \mathbb{R}$

(c) $|ab| = |a||b|$, for all $a, b \in \mathbb{R}$

(d) If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$

(e) $-|a| \leq a \leq |a|$, for all $a \in \mathbb{R}$.

This is a very important inequality:

The triangle inequality: If a, b are any real numbers, then

$$||a|-|b|| \leq |a+b| \leq |a|+|b|$$

Proof: We have, from Theorem 2 (e):

$$-|a| \leq a \leq |a|$$

and

$$-|b| \leq b \leq |b|$$

Theorem 1 (b) implies:

$$-(|a|+|b|) \leq a+b \leq |a|+|b|$$

Hence, Theorem 2 (d) yields: $|a+b| \leq |a|+|b|$. (1)

Now, $|a| = |(a-b)+b| \leq |a-b|+|b|$ (from second part of triangle inequality) and therefore $|a|-|b| \leq |a-b|$. In the same way:

$$|b| = |(b-a)+a| \leq |b-a|+|a| = |a-b|+|a|$$

$$\Rightarrow \quad |b|-|a| \leq |a-b|$$

We have shown that:

$$-|a-b| \leq |a|-|b| \leq |a-b|,$$

and Theorem 2 (d) yields:

$$||a|-|b|| \leq |a-b| \rightarrow (3)$$

Note that (3) holds for any real numbers a, b. In particular it is twe if we replace b by $-b$, hence:

$$||a|-|b|| \leq |a+b| \qquad (4)$$

We conclude, from (1) and (4):

$$||a|-|b|| \leq |a+b| \leq |a|+|b|$$

With similar arguments, we can show:

$$||a|-|b|| \leq |a-b| \leq |a|+|b|,$$ for any real numbers a, b.

<u>Corollary</u>: If $a_1, a_2, \ldots, a_n$ are any n real numbers, then:

$$|a_1+a_2+\cdots+a_n| \leq |a_1|+|a_2|+\cdots+|a_n|.$$

<u>Proof</u>: If $n=2$, the conclusion is precisely the previous theorem. If $n>2$, we use mathematical induction and the fact that:

$$|a_1+a_2+\cdots+a_K+a_{K+1}| = |(a_1+a_2+\cdots+a_K)+a_{K+1}|$$

$$\leq |a_1+a_2+\cdots+a_K| + |a_{K+1}|. \blacksquare$$