Chapter 6

The Riemann-Stieltjes integral for real valued functions on intervals.

<u>Definition</u>: Let $[a,b]$ be a given interval. A partition P of $[a,b]$ is a finite set of points:

$$x_0, x_1, \dots, x_n$$

where:

$$a = x_0 \le x_1 \le \dots \le x_{n-1} \le x_n = b.$$

Let

$$\Delta x_i = x_i - x_{i-1}, \quad i = 1, \dots, n,$$

and $f : [a,b] \to \mathbb{R}$ a bounded function.

Define:

$$M_i := \sup \{ f(x) : x_{i-1} \le x \le x_i \}$$

$$m_i := \inf \{ f(x) : x_{i-1} \le x \le x_i$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \qquad \text{(upper sum)}$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \qquad \text{(lower sum)}$$

$$(1) \qquad \overline{\int_a^b} f \, dx = \inf \{ U(P,f) : P \text{ is a partition} \}$$

$$(2) \qquad \underline{\int_a^b} f \, dx = \sup \{ L(P,f) : P \text{ is a partition} \}$$

(1) and (2) are called the <u>upper</u> and <u>lower</u> <u>Riemann integrals</u> of f over $[a,b]$ respectively.

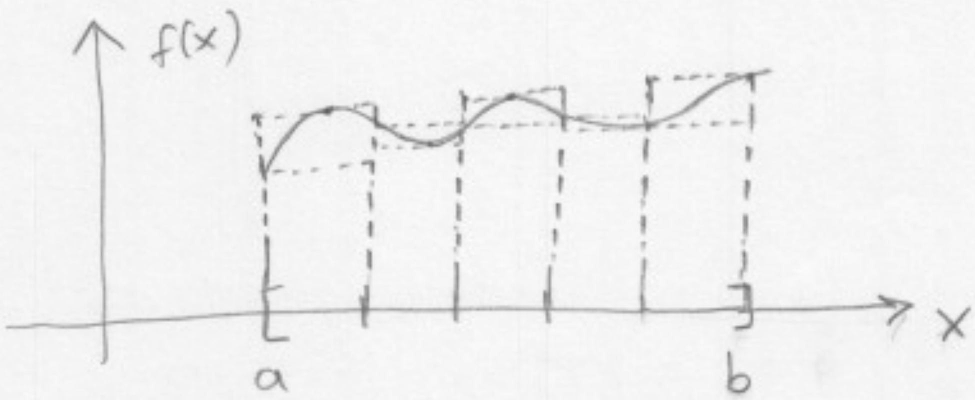
If $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx = \alpha \in \mathbb{R}$, we say that f is Riemann integrable on $[a,b]$.

In this case, we write:

$$f \in \mathcal{R},$$

and we denote the number α as:

$$\int_a^b f\,dx.$$



Since f is bounded:

$$m \le f(x) \le M, \qquad a \le x \le b.$$

Hence, for every partition P, we have:

$$\begin{aligned} m(b-a) = m \sum_{i=1}^{n} \Delta x_i &\le \sum_{i=1}^{n} m_i \Delta x_i \\ &\le \sum_{i=1}^{n} M_i \Delta x_i \\ &\le M \sum_{i=1}^{n} \Delta x_i \\ &= M(b-a) \end{aligned}$$

The previous computation shows that the inf and sup in (1) an (2) always exist. That is, the upper and lower integrals of a bounded function $f: [a,b] \to \mathbb{R}$ always exist. The next question is wheather they are equal or not. We have said, that if the upper and lower integrals of f have the same value, then f is Riemann integrable, and:

$$\int_a^b f\,dx = \underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx.$$

In analysis, there are different types of integrals. For example, in chapter 11, the Lebesgue integral is studied.

Another type of integral is the Riemann-Stieltjes integral. This is a generalization of the Riemann integral as follows:

Definition: Let α be a monotonically increasing function on $[a,b]$. For each partition P, we define:

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

Note that $\Delta\alpha_i \geq 0$. Also, if $\alpha(x) = x$, then $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = x_i - x_{i-1}$, which corresponds to the Riemann integral.

We proceed as before to define the upper and lower integral of f:

$f: [a,b] \to \mathbb{R}$ bounded

$$U(P,f,\alpha) := \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P,f,\alpha) := \sum_{i=1}^{n} m_i \Delta\alpha_i \, ,$$

where M_i, m_i are defined as before.

(3) $\overline{\int_a^b} f\,d\alpha = \inf\{U(P,f,\alpha) : P \text{ is a partition}\}$

(4) $\underline{\int_a^b} f\,dx = \sup\{L(P,f,\alpha) : P \text{ is a partition}\}$

If (3) and (4) are equal we define:

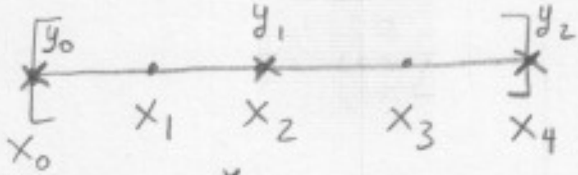
$$\int_a^b f\,d\alpha := \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

This is the Riemann-Stieltjes (or simply the Stieltjes integral) of f with respect to α, over $[a,b]$. We write:

$$f \in \mathcal{R}(\alpha)$$

<u>Note</u>: If $\alpha(x) = x$, then the Riemann-Stieltjes integral reduces to the Riemann integral.

<u>Def</u>: The partition P^* is a refinement of P if $P \subset P^*$



<u>Ex</u>: $P^* = \{x_0, x_1, x_2, x_3, x_4\}$

$P = \{y_0, y_1, y_2\}$

Given two partitions P_1 and P_2 we say that P^* is their common refinement if:

$$P^* = P_1 \cup P_2 .$$

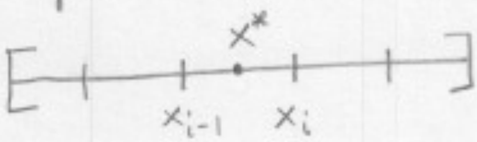
<u>Theorem 6.4</u>: If P^* is a refinement of P, then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \qquad (1)$$

and

$$U(P^*, f, \alpha) \leq U(P, f, \alpha) \qquad (2)$$

<u>Proof</u>: Assume first that P^* contains just one more point than P.



Suppose that:

$$x_{i-1} < x^* < x_i$$

Let:

$$w_1 = \inf\{f(x) ; x_{i-1} \leq x \leq x^*\}$$
$$w_2 = \inf\{f(x) : x^* \leq x \leq x_i\}$$

Since $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ we have:

$$w_1 \geq m_i \quad , \quad w_2 \geq m_i$$

$$\begin{aligned} L(P^*, f, \alpha) - L(P, f, \alpha) &= w_1 [\alpha(x^*) - \alpha(x_{i-1})] \\ &\quad + w_2 [\alpha(x_i) - \alpha(x^*)] \\ &\quad - m_i [\alpha(x_i) - \alpha(x_{i-1})] \\ &\geq m_i [\alpha(x_i) - \alpha(x_{i-1})] \\ &\quad - m_i [\alpha(x_i) - \alpha(x_{i-1})] \\ &\geq 0 \end{aligned}$$

If P^* contains K points more than P, we repeat this reasoning K times, which proves (1). In a similar way we obtain (2).

Theorem 6.5:

$$\underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha$$

Proof: Let P_1, P_2 be two partitions of $[a,b]$. Consider the common refinement $P^* = P_1 \cup P_2$.

Then:

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$$

$$\Rightarrow \quad L(P_1, f, \alpha) \leq U(P_2, f, \alpha) \quad (1)$$

Then, (1) holds for all partitions P_1 and P_2. We fix P_2 and take the sup in (1) of all lower sums. Since $U(P_2, f, \alpha)$ is an upper bound, from definition of sup, we get

$$\underline{\int_a^b} f \, d\alpha \leq U(P_2, f, \alpha) \qquad (2)$$

Since (2) holds for all partitions P_2, we take the inf in (2) over all upper sums. Since $\underline{\int_a^b} f\, d\alpha$ is a lower bound, from the definition of inf, we get:

$$\underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha . \quad \blacksquare$$

<u>Theorem 6.6</u>: $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if $\forall \varepsilon > 0$, $\exists P$ s.t:

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \qquad (C\,I)$$

<u>Proof</u>: For every partition P we have:

$$L(P,f,\alpha) \le \underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha \le U(P,f,\alpha) \qquad (*)$$

$(\Longleftarrow)$: Let $\varepsilon > 0$. From the hypothesis, $\exists P$ s.t:

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon . \qquad (**)$$

From $(*)$ and $(**)$ it follows that:

$$0 \le \overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\, d\alpha < \varepsilon ,$$

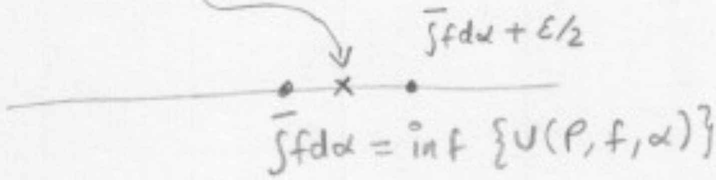
and since this is true for every $\varepsilon > 0$, we conclude that $\overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha = 0$. That is:

$$\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha , \quad \text{and } f \in \mathcal{R}(\alpha).$$

($\Longrightarrow$) Let $f \in \mathcal{R}(\alpha)$, and let $\varepsilon > 0$.

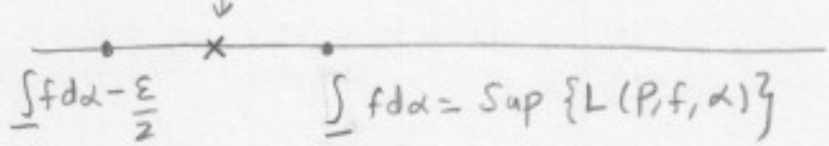
By definition of inf, $\exists P_2$ such that

$$U(P_2, f, \alpha) < \overline{\int_a^b} f \, d\alpha + \frac{\varepsilon}{2} \qquad (A)$$



By definition of sup, $\exists P_1$ such that:

$$L(P_1, f, \alpha) > \underline{\int_a^b} f \, d\alpha - \frac{\varepsilon}{2} \qquad (B)$$



Let $P = P_1 \cup P_2$

We have, from (A) and (B):

$$U(P, f, \alpha) \leq U(P_2, f, \alpha) < \overline{\int_a^b} f \, d\alpha + \frac{\varepsilon}{2} \; ; \text{ by } (A)$$

$$< L(P_1, f, \alpha) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \; ; \text{ by } (B)$$

$$= L(P_1, f, \alpha) + \varepsilon$$

therefore:

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon . \blacksquare$$