Recall the Criterion for integrability (denoted as CI):

$f \in \mathcal{R}(\alpha)$ on $[a,b] \Longleftrightarrow \forall \varepsilon > 0, \exists P$ s.t.

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon.$$

Theorem 6.7:

(a) If (CI) holds for some P and some ε, then (CI) holds (with the same ε) for every refinement of P.

(b) If (CI) holds for $P = \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$ then:

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon.$$

(c) If $f \in \mathcal{R}(\alpha)$ and the hypothesis of (b) holds, then:

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon$$

Proof: (a) We have:

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Let $P^* \supset P$ be any refiment of P

We have:

$$L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha)$$

$$\Longrightarrow \quad U(P^*,f,\alpha) - L(P^*,f,\alpha) < \varepsilon.$$

(b) $$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i \leq \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$
$$= U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

(c) $$L(P, f, \alpha) \leq \sum_{i=1}^{n} f(t_i) \Delta\alpha_i \leq U(P, f, \alpha)$$

We also have:
$$L(P, f, \alpha) \leq \int_a^b f \, d\alpha \leq U(P, f, \alpha)$$

Therefore:
$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon \quad . \; \blacksquare$$

<u>Theorem 6.8</u>: If f is continuous on $[a, b]$, then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

<u>Proof</u>: Let $\varepsilon > 0$. Choose $\eta > 0$ so that:
$$(\alpha(b) - \alpha(a))\eta < \varepsilon \qquad (1)$$

Since $[a, b]$ is compact and $f: [a, b] \to \mathbb{R}$ is continuous, then is uniformly continuous on $[a, b]$.

Then, $\exists \delta > 0$ s.t:
$$x, t \in [a, b], \; |x - t| < \delta \implies |f(x) - f(t)| < \eta \quad (*)$$

Let P be a partition satisfying:

$$\Delta x_i < \delta .$$

By $(*)$ it follows that:

Since f is continuous on $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$, and $[x_{i-1}, x_i]$ is compact, we have:

$$M_i = f(s_i), \quad m_i = f(t_i), \text{ for some } s_i, t_i \in [x_{i-1}, x_i]$$

(Recall that a continuous function on a compact set attains its maximum and minimum at the set).

Since $|s_i - t_i| < \delta \Rightarrow |f(s_i) - f(t_i)| = M_i - m_i < \eta$, where we have used $(*)$

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^{n} (M_i - m_i)\, \Delta\alpha_i \\ &< \eta \sum_{i=1}^{n} \Delta\alpha_i \\ &= \eta\, [\alpha(b) - \alpha(a)] < \varepsilon, \text{ from } (1) \end{aligned}$$

We have proved:

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon .$$

From the criterion of integrability we conclude:

$$f \in \mathcal{R}(\alpha). \quad \blacksquare$$

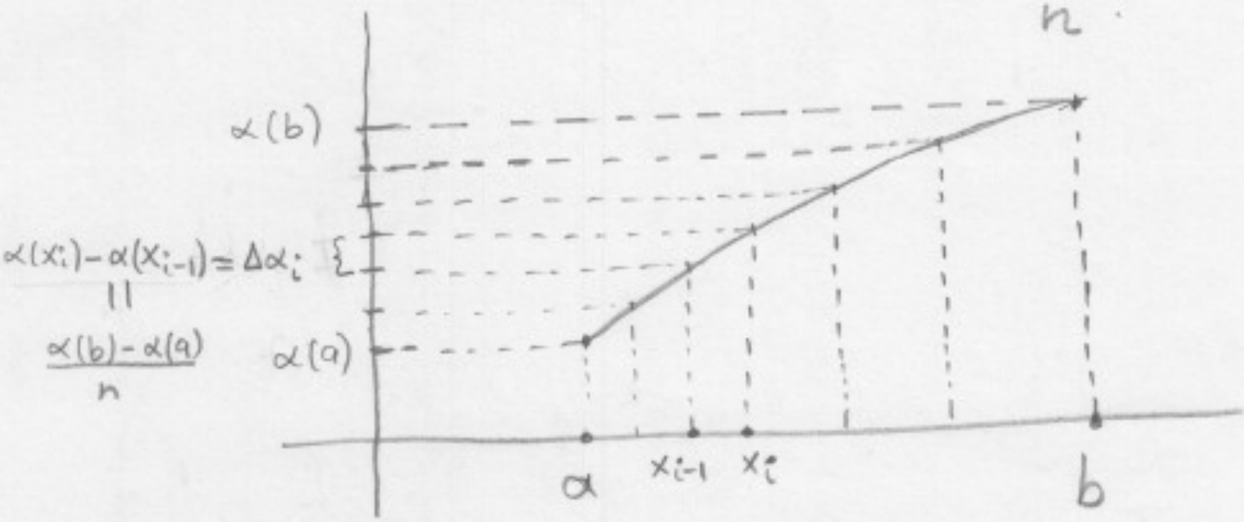
Theorem 6.9: If f is monotonic on $[a,b]$ and if α is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$.

Proof:

Remark: Recall that α is always monotonically increasing.
Recall also that if $\alpha(x) = x$, the Riemann-Stieltjes integral is just the Riemann integral.
Clearly, $\alpha(x) = x$ is continuous on $[a,b]$.
Hence, if f is monotonic on $[a,b]$, f is Riemann integral ($f \in \mathcal{R}$).

Let $\varepsilon > 0$. Since α is continuous, from Theorem 4.23 it follows that, for each integer n, there exists a partition P_n satisfying:

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \frac{\alpha(b) - \alpha(a)}{n} \qquad (1)$$



• We consider the case f is monotonically increasing (the same type of argument applies if f is monotonically decreasing).

We have:

$$M_i = f(x_i), \quad m_i = f(x_{i-1}), \; i = 1, \dots, n \qquad (2)$$

We compute:

$$\begin{aligned} U(P_n, f, \alpha) - L(P_n, f, \alpha) &= \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\ &= \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)], \end{aligned}$$

where we have used (1) and (2).

Since $\frac{(\alpha(b)-\alpha(a))(f(b)-f(a))}{n} \to 0$ as $n \to \infty$, $\exists N$ s.t.:

$$\frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] < \varepsilon.$$

Then:

$$U(P_N, f, \alpha) - L(P_N, f, \alpha) < \varepsilon.$$

The criterion of integrability (CI) yields:

$$f \in \mathcal{R}(\alpha).$$

- <u>Theorem 6.10</u>; Suppose f is bounded on $[a,b]$ and suppose that f has only finitely many points of discontinuity on $[a,b]$. Suppose also that α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.

<u>Proof</u>:

<u>Remark</u>: Since $\alpha(x) = x$ is continuous on $[a,b]$, this theorem says that if f is bounded and has only finitely many points of discontinuity on $[a,b]$ then $f \in \mathcal{R}$.

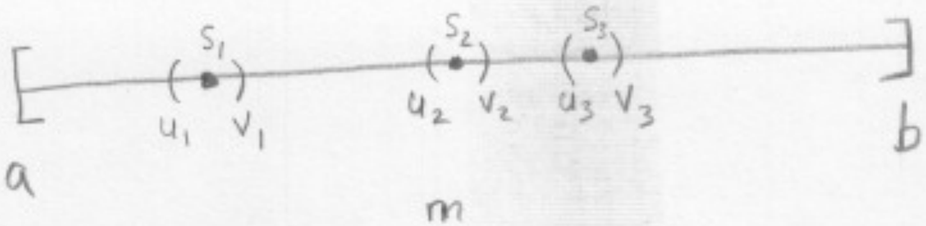
<u>Proof</u>: Let $\varepsilon > 0$. Since f is bounded, $\exists M$ s.t.

$$\boxed{|f(x)| \leq M, \quad \forall x \in [a,b]. \qquad (***)}$$

Let E be the set of points of discontinuity of f, say $E = \{s_1, \dots, s_n\}$. Since E is finite and α is continuous at s_i, we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset [a,b]$, $s_j \in [u_j, v_j]$, $j = 1, \dots, m$ and

$$\boxed{\sum_{j=1}^{m} \alpha(v_j) - \alpha(u_j) < \varepsilon. \qquad (*)}$$

We can place the intervals $[u_j, v_j]$ in such a way that every point of $E \cap (a,b)$ lies in the interior of the interval.

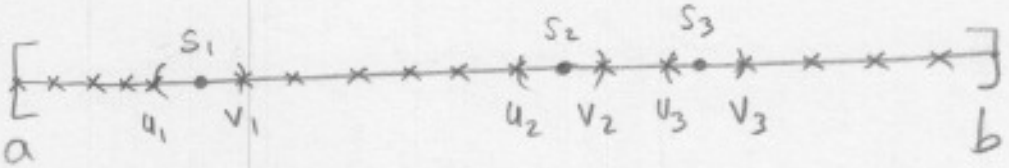


Let $K = [a,b] \setminus \bigcup_{i=1}^{m} (u_i, v_j)$, K is compact (closed and bounded)

Since f is continuous on $K \Rightarrow f$ is uniformly continuous. Therefore:

$$\boxed{\exists \delta > 0 \text{ s.t. } s,t \in K,\ |s-t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon} \quad (**)$$

Form a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ as follows: each u_j belongs to P. Also, each v_j belongs to P. $P \cap (u_j, v_j) = \emptyset$, $j = 1, \dots, m$. Also, we choose $\{x_0, \dots, x_n\}$ such that $\Delta x_i = x_i - x_{i-1} < \delta$, if x_{i-1} is not one of the u_j.



$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i$$

$$= \sum_{\substack{\text{over the} \\ (u_i, v_i)}} (M_i - m_i)\,\Delta\alpha_i + \sum_{\substack{\text{over the} \\ \text{rest}}} (M_i - m_i)\,\Delta\alpha_i$$

$$\le \underbrace{2}_{\text{by } (***)} \underbrace{M}_{\text{by } (*)} \varepsilon + \underbrace{\varepsilon}_{\text{by } (**)} [\alpha(b) - \alpha(a)]$$

Since ε is arbitrary, (CI) $\Rightarrow f \in \mathcal{R}(\alpha)$. ∎

<u>Theorem 6.11</u>: Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \le f(x) \le M$, $x \in [a,b]$. Let ϕ continuous on $[m, M]$ and let:

$$h(x) = \phi(f(x)), \quad x \in [a,b]$$

Then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

<u>Proof</u>: Let $\varepsilon > 0$

Since ϕ is uniformly continuous on $[m, M]$, $\exists \delta$, $\delta < \varepsilon$ s.t:

$$\boxed{s, t \in [m, M], \ |s - t| < \delta \Rightarrow |\phi(s) - \phi(t)| < \varepsilon.} \quad (1)$$

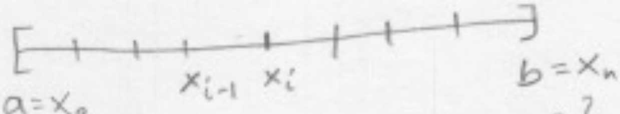
Since $f \in \mathcal{R}(\alpha) \Rightarrow \exists P$ s.t:

$$\boxed{U(P, f, \alpha) - L(P, f, \alpha) < \delta^2.} \quad (*)$$

Let:

$M_i^* = \sup\{h(x) : x_{i-1} \le x \le x_i\}$

$m_i^* = \inf\{h(x) : x_{i-1} \le x \le x_i\}$, $K = \sup\limits_{m \le t \le M} |\phi(t)|$



$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$

$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$

Let $A = \{i : M_i - m_i < \delta\}$

$B = \{i : M_i - m_i \ge \delta\}$

$P = \{x_0, x_1, \dots, x_n\}$

$$\begin{tikzcd} {[x_{i-1}, x_i]} \arrow[r, "f"] \arrow[rr, bend right, "h"'] & {[m_i, M_i]} \arrow[r, "\phi"] & \mathbb{R} \end{tikzcd}$$

We need to estimate $U(P, h, \alpha) - L(P, h, \alpha)$.

$$f([x_{i-1}, x_i]) \subset [m_i, M_i] \quad (2)$$

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i\in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum_{i\in B} (M_i^* - m_i^*)\Delta\alpha_i$$

Now, from (1) and (2):

$$\sum_{i\in A} (M_i^* - m_i^*)\Delta\alpha_i < \varepsilon \sum_{i\in A} \Delta\alpha_i, \quad \text{since } M_i - m_i < \delta$$
$$= \varepsilon\,[\alpha(b) - \alpha(a)]$$

$$\sum_{i\in B} (M_i^* - m_i^*)\Delta\alpha_i \le 2K \sum_{i\in B} \Delta\alpha_i \;; \quad \text{since } K = \sup_{m\le t\le M} |\phi(t)|$$

In this case:

$$M_i - m_i \ge \delta \Rightarrow \delta \sum_{i\in B} \Delta\alpha_i \le \sum_{i\in B} (M_i - m_i)\Delta\alpha_i \le \delta^2; \text{ by } (*)$$
$$\Rightarrow \sum_{i\in B} \Delta\alpha_i \le \delta$$

Hence:

$$\sum_{i\in B} (M_i^* - m_i^*)\Delta\alpha_i \le 2K\delta$$

We have shown:

$$U(P,h,\alpha) - L(P,h,\alpha) < \varepsilon\,[\alpha(b) - \alpha(a)] + 2K\delta$$
$$< \varepsilon\,[\alpha(b) - \alpha(a)] + 2K\varepsilon$$
$$= \varepsilon\,(\alpha(b) - \alpha(a) + 2K),$$

and this is twe for every ε. In particular, if we take $\dfrac{\varepsilon}{\alpha(b) - \alpha(a) + 2K}$ instead of ε at the beginning of the proof we get $U(P,h,\alpha) - L(P,h,\alpha) < \varepsilon$.

Hence, $(CI) \Rightarrow h \in \mathcal{R}(\alpha)$.

<u>Remark</u>: In Chapter 11 (Theorem 11.33), using measure theory, the following theorem is proven:

<u>Thm</u>: Suppose f is bounded on $[a,b]$. Then $f \in \mathcal{R}$ on $[a,b]$ if and only if f is continuous almost everywhere on $[a,b]$.

Properties of the integral:

<u>Theorem 6.12</u>

(a) Let $f_1 \in \mathcal{R}(\alpha)$, $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$. Then:

$f_1 + f_2 \in \mathcal{R}(\alpha)$, $cf \in \mathcal{R}(\alpha)$, $c \in \mathbb{R}$, and:

$$\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$$

$$\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$$

(b) If $f_1(x) \le f_2(x)$ on $[a,b]$ then:

$$\int_a^b f_1\,d\alpha \le \int_a^b f_2\,d\alpha$$

(c) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $a<c<b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and $f \in \mathcal{R}(\alpha)$ on $[c,b]$.

Moreover:

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha$$

(d) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $|f(x)| \le M$ on $[a,b]$ then:

$$\left| \int_a^b f\, d\alpha \right| \le \int_a^b |f(x)|\, d\alpha \text{ ; see Theorem 6.13}$$
$$\le M \int_a^b d\alpha$$
$$= M\,(\alpha(b) - \alpha(a))$$

(e) If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$ then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and:

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$

If $f \in \mathcal{R}(\alpha)$ and $c \in \mathbb{R}$, $c > 0$, then $f \in \mathcal{R}(c\alpha)$ and:

$$\int_a^b f\, d(c\alpha) = c \int_a^b f\, d\alpha.$$

Theorem 6.13 : If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$ then

(a) $fg \in \mathcal{R}(\alpha)$

(b) $|f| \in \mathcal{R}(\alpha)$ and $\left|\int_a^b f\, d\alpha\right| \leq \int_a^b |f|\, d\alpha$.

Proof :

(a) Let $\phi(t) = t^2$. Then $(\phi \circ f)(x) = \phi(f(x)) = (f(x))^2$

Using Theorem 6.11 we obtan $f^2 \in \mathcal{R}(\alpha)$.

$$4fg = (f+g)^2 - (f-g)^2$$

Theorem 6.12 $\Rightarrow$ $f+g \in \mathcal{R}(\alpha)$, $f-g \in \mathcal{R}(\alpha)$, $(f+g)^2 \in \mathcal{R}(\alpha)$, $(f-g)^2 \in \mathcal{R}(\alpha)$ and finally $fg \in \mathcal{R}(\alpha)$.

(b) Let $\phi(t) = |t|$. Again, we form the composition:

$$(\phi \circ f)(x) = \phi(f(x)) = |f(x)|$$

theorem 6.11 gives $|f| \in \mathcal{R}(\alpha)$.

Now:

$$\left|\int_a^b f\, d\alpha\right| = c\int_a^b f\, d\alpha = \int_a^b cf\, d\alpha \leq \int_a^b |f|\, d\alpha,$$

$c = 1$ or $c = -1$

because $cf \leq |f|$, and using Theorem 6.12 (b). ∎