Theorem 6.19 (Change of Variables).

Suppose:

- $\varphi : [A, B] \to [a, b]$

  φ is a strictly increasing continuous function

- α monotonically increasing on $[a, b]$
- $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

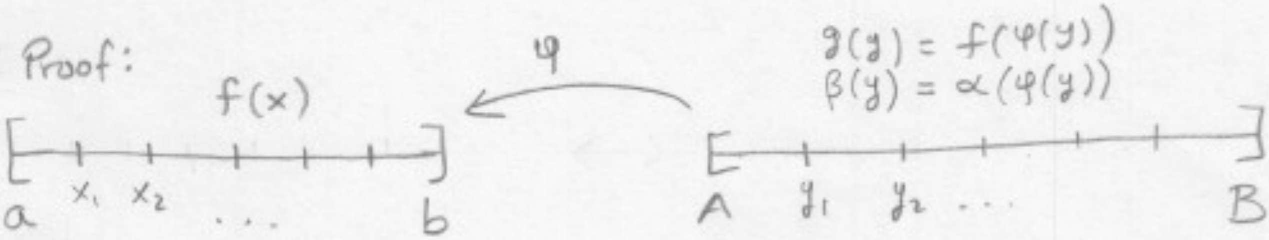
Define:

$$\beta, g : [A, B] \to \mathbb{R}$$
$$\beta(y) = \alpha(\varphi(y)), \quad g(y) = f(\varphi(y))$$

Then:

$$g \in \mathcal{R}(\beta) \quad \text{and} \quad \int_A^B g \, d\beta = \int_a^b f \, d\alpha$$

Sketch of Proof:



$$P = \{x_0, x_1, \dots, x_n\} \qquad Q = \{y_0, \dots, y_n\}$$

$$x_i = \varphi(y_i), \quad i = 0, 1, 2, \dots, n$$

$$U(Q, g, \beta) = U(P, f, \alpha), \quad L(Q, g, \beta) = L(P, f, \alpha)$$

Let $\varepsilon > 0$:

Since $f \in \mathcal{R}(\alpha) \Rightarrow \exists P$ s.t.:

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

$$\Rightarrow \quad U(Q, g, \beta) - L(Q, g, \beta) < \varepsilon \quad \Rightarrow g \in \mathcal{R}(\beta)$$

- To see that $\int_a^b f\,d\alpha = \int_A^B g\,d\beta$ we note that $\forall \varepsilon > 0$, $\exists P$ s.t:

$$U(P, f, \alpha) - \int_a^b f\,d\alpha < \varepsilon$$

and $\int_a^b f\,d\alpha - L(P, f, \alpha) < \varepsilon$,

and from here, since $U(P, f, \alpha) = U(Q, g, \beta)$ and $L(P, f, \alpha) = L(Q, g, \beta)$, it follows that:

$$\int_a^b f\,d\alpha = \int_A^B g\,d\beta. \quad \blacksquare$$

Ex: Let $\alpha(x) = x$. Then $\beta(y) = \varphi(y)$. Assume that $\varphi' \in \mathcal{R}$ on $[A, B]$. From Theorem 6.19:

$$\int_a^b f\,dx = \int_A^B g\,d\varphi$$

$$= \int_A^B f(\varphi(y))\,d\varphi$$

$$= \int_A^B f(\varphi(y))\,\varphi'(y)\,dy; \text{ by Theorem 6.17}$$

We conclude:

$$\boxed{\int_a^b f\,dx = \int_A^B f(\varphi(y))\,\varphi'(y)\,dy}$$

<u>Ex</u>: Compute $\int_0^1 f(x)\,dx$,

$f(x) = \sqrt{1-x^2}$

We use:

$$\int_a^b f(x)\,dx = \int_A^B f(\varphi(y))\varphi'(y)\,dy$$

$$[0,1] \xleftarrow{\ \varphi\ } [0, \tfrac{\pi}{2}]$$

$$\varphi(y) = \sin y$$

$$\int_0^1 \underbrace{\sqrt{1-x^2}}_{f(x)}\,dx = \int_0^{\pi/2} \underbrace{\sqrt{1-\sin^2 y}}_{f(\varphi(y))}\ \underbrace{\cos y}_{\varphi'(y)}\,dy$$

It is now easy to integrate $\int_0^{\pi/2} \cos^2 y\,dy$.

<u>Def</u>: Let $f_1, \dots, f_k$ be real functions on $[a,b]$, and let $\vec{f} = (f_1, \dots, f_k)$; that is, $f: \mathbb{R} \to \mathbb{R}^k$.

$\vec{f} \in \mathcal{R}(\alpha)$ means $f_j \in \mathcal{R}(\alpha)$, $j = 1, \dots, k$. In this case:

$$\int_a^b \vec{f}\,d\alpha = \left(\int_a^b f_1\,d\alpha, \dots, \int_a^b f_k\,d\alpha \right).$$

We also have:

<u>Thm 6.24</u>. Let $\vec{f}: [a,b] \to \mathbb{R}^k$, $\vec{F}: [a,b] \to \mathbb{R}^k$. If $\vec{f} \in \mathcal{R}$ on $[a,b]$ and if $\vec{F}' = \vec{f}$, then

$$\int_a^b \vec{f}(t)\,dt = \vec{F}(b) - \vec{F}(a). \quad \boxtimes$$

<u>Theorem 6.25</u>: Let $\vec{f}:[a,b]\to\mathbb{R}^k$, $\vec{f}\in\mathcal{R}(\alpha)$.
Then:

$$|\vec{f}|\in\mathcal{R}(\alpha) \quad \text{and} \quad \left|\int_a^b \vec{f}\,d\alpha\right| \le \int_a^b |\vec{f}|\,d\alpha$$

<u>Proof</u>: $|\vec{f}| = (f_1^2+\cdots+f_k^2)^{1/2}$

$f_i^2\in\mathcal{R}(\alpha)$, $i=1,\dots,k$. This is because of the chain rule Theorem 6.11. If $\phi(t)=t^2$, we note that $(\phi\circ f_i)(t) = f_i(t)^2$.

$$\Rightarrow \sum_{i=1}^{k} f_i^2 \in\mathcal{R}(\alpha)$$

Since $t\mapsto\sqrt{t}$ is continuous, another application of Theorem 6.11 yields $|\vec{f}|\in\mathcal{R}(\alpha)$.

Let $\vec{y}=(y_1,\dots,y_k)$, $y_i=\int_a^b f_i\,d\alpha$

$$\Rightarrow \vec{y}=\int_a^b \vec{f}\,d\alpha$$

We note:

$$|\vec{y}|^2 = \sum_{i=1}^{k} y_i^2 = \sum_{i=1}^{k} y_i\int_a^b f_i\,d\alpha$$

$$= \int_a^b\left(\sum_{i=1}^{k} y_i f_i(t)\right)d\alpha$$

$$\le \int_a^b |\vec{y}|\,|\vec{f}(t)|\,d\alpha; \text{ by Schwartz inequality}$$

$$\Rightarrow |\vec{y}|^2 \le |\vec{y}|\int_a^b |\vec{f}|\,d\alpha$$

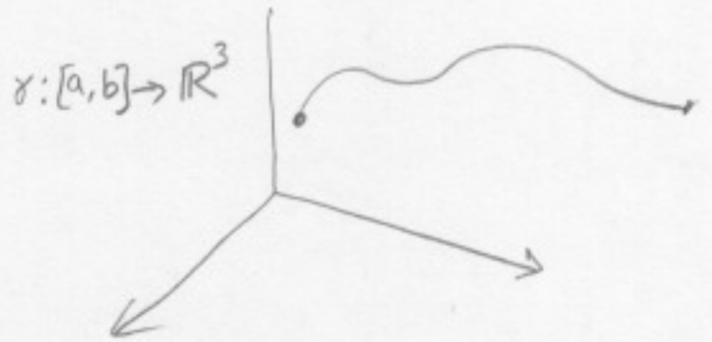
$$\Rightarrow |\vec{y}| \le \int_a^b |\vec{f}|\,d\alpha; \text{ since } |\vec{y}|\ne 0.$$

$$\Rightarrow \left|\int_a^b \vec{f}\,d\alpha\right| \le \int_a^b |\vec{f}|\,d\alpha. \quad \blacksquare$$

The lenght of a curve.

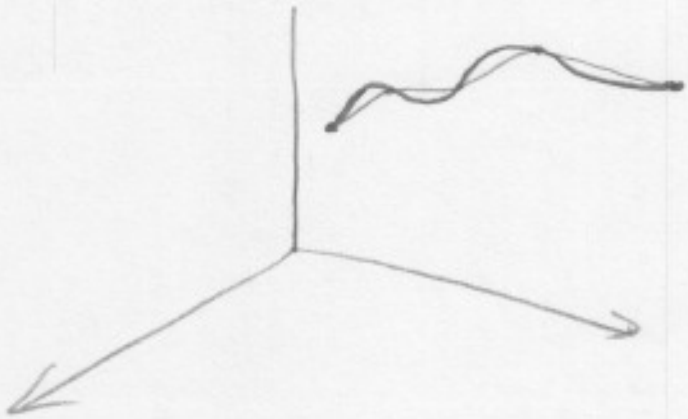
Def: A continuous mapping $\gamma: [a,b] \to \mathbb{R}^k$ is called a curve in $\mathbb{R}^k$. If γ is one-to-one, γ is called an arc. If $\gamma(a) = \gamma(b)$, γ is said to be a closed curve.

Note: Different functions γ may yield the same range in $\mathbb{R}^k$.



Define:

$$L(P,\gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$



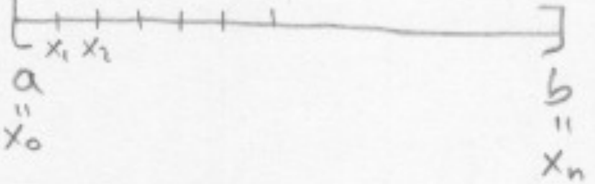
We compute the length by approximating with lines.

Define: $L(\gamma) = \sup L(P, \gamma)$

Theorem 6.27: If γ' is continuous on $[a,b]$, then:

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt .$$

Proof: Let P be a partition of $[a,b]$



Using the fundamental theorem of calculus:

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\, dt \right| \leq \int_{x_{i-1}}^{x_i} |\gamma'(t)\, dt$$

$$\Rightarrow \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)\, dt$$

$$\Rightarrow \quad L(P,\gamma) \leq \int_a^b |\gamma'(t)\, dt, \text{ for every partition } P.$$

Since $L(\gamma) = \sup\{L(P,\gamma)\}$ we obtain:

$$\Rightarrow \boxed{L(\gamma) \leq \int_a^b |\gamma'(t)|\, dt} \quad (1)$$

Let $\varepsilon > 0$. Since γ' is continuous on $[a,b]$ then γ' is uniformly continuous. Then, $\exists\, \delta > 0$ s.t.

$$|s-t| < \delta \implies |\gamma'(s) - \gamma'(t)| < \varepsilon . \quad (2)$$

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ with $\Delta x_i < \delta$. If $x_{i-1} \leq t \leq x_i$, it follows that:

$$|\gamma'(t)-\gamma'(x_i)|<\varepsilon$$

$$\Rightarrow \quad |\gamma'(t)| \le |\gamma'(x_i)|+\varepsilon \; ; \quad \text{since: } ||a|-|b|| \le |a-b|$$

$$\Rightarrow$$

$$\int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \le \int_{x_{i-1}}^{x_i} |\gamma'(x_i)|\,dt + \int_{x_{i-1}}^{x_i} \varepsilon \; dt$$

$$= \left| \int_{x_{i-1}}^{x_i} \gamma'(x_i)\,dt \right| + \varepsilon \Delta x_i, \quad \text{because } \gamma'(x_i) \text{ is a constant vector}$$

$$= \left| \int_{x_{i-1}}^{x_i} (\gamma'(x_i)-\gamma'(t))\,dt + \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right| + \varepsilon \Delta x_i$$

$$\le \int_{x_{i-1}}^{x_i} |\gamma'(x_i)-\gamma'(t)|\,dt + \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right| + \varepsilon \Delta x_i$$

$$\leqslant \varepsilon \Delta x_i + \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right| + \varepsilon \Delta x_i, \text{ by } (2)$$

$$= |\gamma(x_i)-\gamma(x_{i-1})| + 2\varepsilon \Delta x_i, \quad \text{by fundamental theorem of calculus}$$

$$\Rightarrow \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \le \sum_{i=1}^{n} |\gamma(x_i)-\gamma(x_{i-1})| + 2\varepsilon(b-a)$$

$$\Rightarrow \quad \int_a^b |\gamma'(t)|\,dt \le L(P,\gamma) + 2\varepsilon(b-a)$$

$$\Rightarrow \quad \int_a^b |\gamma'(t)|\,dt \le L(\gamma) + 2\varepsilon(b-a)$$

Since ε is arbitrary:

$$\boxed{\int_a^b |\gamma'(t)|\,dt \leq L(\gamma)} \quad (3)$$

From (1) and (3) we conclude:

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt. \quad \blacksquare$$