Sequences and series of functions.

Let E be any set. Consider:

$$f : E \to \mathbb{R}.$$

Many theorems in this chapter extend to vector-valued functions or functions into general metric spaces.

<u>Definition</u>: Let $\{f_n\}$, $n = 1, 2, \dots$, be a sequence of functions defined on a set E, and suppose that:

$\{f_n(x)\}$ is a convergent sequence of real numbers, for every $x \in E$.

In this case we define:

$$f(x) := \lim_{n \to \infty} f_n(x), \quad x \in E. \quad (*)$$

If $(*)$ holds we say:

$$f_n \to f \text{ pointwise on } E.$$

<u>Question</u>: Suppose $f_n \to f$ pointwise and f_n is continuous, $n = 1, 2, \dots$. Is f continuous?

Recall that f is continuous at x if

$$\lim_{t \to x} f(t) = f(x)$$

We have:

$$\left.\begin{aligned} \lim_{t \to x} f(t) &= \lim_{t \to x} \lim_{n \to \infty} f_n(t) \\ f(x) &= \lim_{n \to \infty} \lim_{t \to x} f_n(t) \end{aligned}\right] \to$$

Are these two expressions equal?
In general, not true.

Ex: Let

$$f_n(x) = \frac{1}{n} \sin(nx + n), \quad x \in \mathbb{R}.$$

$$0 \le |f_n(x)| \le \frac{1}{n}$$

$$\therefore \lim_{n\to\infty} f_n(x) = 0$$

$$\therefore \quad f_n \to 0 \text{ pointwise on } \mathbb{R}$$

Ex: Let $f_n(x) = x^n, \quad x \in [0,1]$.

$f_n \to f$ pointwise, where:

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$$

Note in this example that f_n is continuous, for all $n = 1, 2, 3, \ldots$. However, f is discontinuous at $x = 1$.

Ex: Let $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}, \quad x \in \mathbb{R}, \ n = 1, 2, \ldots$

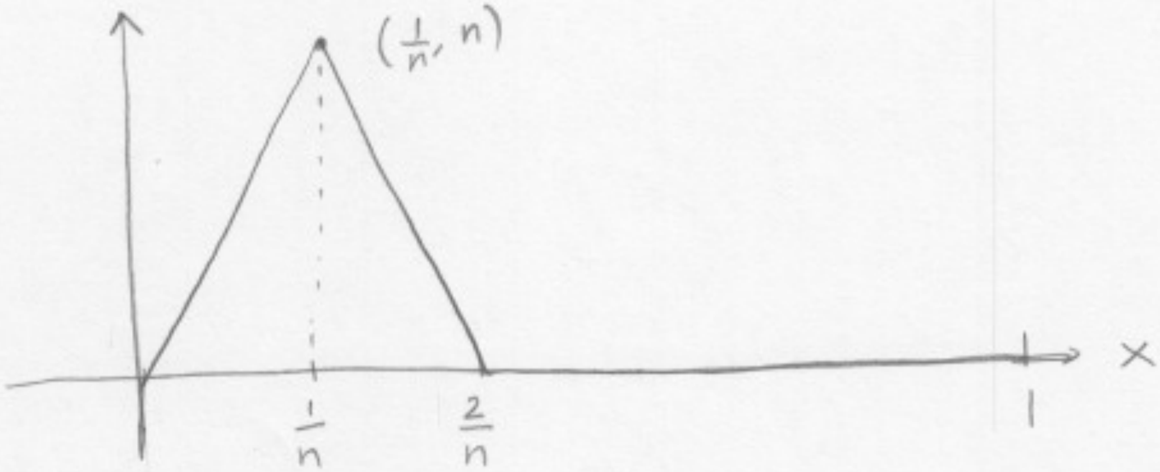
$f_n \to f$ pointwise with $f(x) = 0, \ \forall x \in \mathbb{R}$.

$$f_n'(x) = \sqrt{n} \cos nx, \quad f'(x) = 0, \ x \in \mathbb{R}$$

Note that $f_n \to f$ pointwise but $f_n' \not\to f'$ pointwise since $f_n'(0) = \sqrt{n} \to +\infty$.

<u>Ex</u>: Let $f_n : [0,1] \to \mathbb{R}$, $n \geq 2$,

$$f_n(x) = \begin{cases} n^2 x, & 0 \leq x \leq \frac{1}{n} \\ -n^2\left(x - \frac{2}{n}\right), & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \frac{2}{n} \leq x \leq 1 \end{cases}$$



f_n is continuous, $n = 2, \ldots$

$f_n \to 0$ pointwise on $[0,1]$

$$\int_0^1 f_n(x)\,dx = \frac{1}{2} \cdot \left(\frac{2}{n} \cdot n\right) = 1.$$

This example shows that, with $f \equiv 0$, :

$$f_n \to f \quad \text{pointwise on } [0,1]$$

but:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \neq 0 = \int_0^1 f(x)\,dx$$

Uniform Convergence

<u>Def</u>: The sequence $\{f_n\}$ converges uniformly on E to f if $\forall \varepsilon > 0, \exists N$ s.t:

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N, \quad \forall x \in E$$

<u>Note</u>: Uniform convergence means that, for each $\varepsilon > 0$, we can find <u>one</u> integer N which will do <u>for all</u> $x \in E$.

<u>Def</u>: We say that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of partial sums:

$$S_n(x) = \sum_{i=1}^{n} f_i(x)$$

converges uniformly on E

<u>Remarks on Series</u>: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We define:

$$S_n = \sum_{k=1}^{n} a_k$$

The sequence $\{S_n\}_{n=1}^{\infty}$ is called the sequence of partial sums. If $\{S_n\}$ converges to $s \in \mathbb{R}$, we say that the series $\sum_{n=1}^{\infty} a_n$ converges, and write:

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the sum of the series; but it should be clearly understood that s is the limit of a sequence of sums, and is not obtained simply by addition. If $\{s_n\}$ diverges, the series is said to diverge.

Theorem 3.22 : $\sum_{n=1}^{\infty} a_n$ converges $\iff \forall \varepsilon > 0, \exists N$ s.t.: $\left| \sum_{k=n}^{m} a_k \right| < \varepsilon$, if $m \geq n \geq N$.

In particular, by taking $m = n$ we get $|a_n| < \varepsilon$, $\forall n \geq N$. That is, if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n \to \infty} a_n = 0$. The opposite is not true. For example, $\frac{1}{n} \to 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Note that Theorem 3.22 follows from Theorem 3.11: "A sequence converges in $\mathbb{R}^k \iff$ it is a Cauchy sequence".

Hence:

$$
\begin{aligned}
\{S_n\} \text{ converges} &\iff \{S_n\} \text{ is Cauchy} \\
&\iff \forall \varepsilon > 0, \exists N \text{ s.t } |S_n - S_m| < \varepsilon \quad \forall m \geq n \geq N. \\
&\iff \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right| < \varepsilon \\
&\iff \left| \sum_{k=n}^{m} a_k \right| < \varepsilon \quad \forall m \geq n \geq N
\end{aligned}
$$

The following is called a "comparison test":

Theorem 3.25:

(a) If $|a_n| \le c_n$ for $n \ge N_0$, where N_0 is some fixed integer, and if $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

(b) If $a_n \ge d_n \ge 0$ for $n \ge N_0$, and if $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges. Part (b) applies only to series of nonnegative terms a_n.

Proof: Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} c_n$ converges, $\exists N \ge N_0$ such that:

$$\sum_{k=n}^{m} c_k < \varepsilon \quad \forall m \ge n \ge N$$

$$\Rightarrow \left| \sum_{k=n}^{m} a_k \right| \le \sum_{k=n}^{m} |a_k| \le \sum_{k=n}^{m} c_k < \varepsilon, \text{ and (a)}$$

follows.

(b) follows from (a), for if $\sum_{n=1}^{\infty} a_n$ converges, so must $\sum_{n=1}^{\infty} d_n$. ∎

Theorem 3.26 If $0 \le x < 1$, then:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \ge 1$, the series diverges.

Proof:

$$S_n = 1 + x + x^2 + \cdots + x^n$$

$$\Rightarrow \quad xS_n = x + x^2 + x^3 + \cdots + x^{n+1}$$

$$\Rightarrow \quad S_n - xS_n = 1 - x^{n+1}$$

$$S_n(1-x) = 1 - x^{n+1}$$

$$S_n = \frac{1-x^{n+1}}{1-x}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1-0}{1-x} = \frac{1}{1-x}, \text{ if } x \neq 1$$

For $x = 1$, we have $1+1+1+\cdots$, which clearly diverges.

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$

We also have the following:

Theorem 3.28: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem 7.8 :

$f_n \to f$ uniformly on $E \iff \forall \varepsilon > 0, \exists N$ s.t :

$$|f_n(x) - f_m(x)| < \varepsilon, \ \forall x \in E, \ n, m \geq N$$

(This condition is called the Cauchy condition (CC)).

Proof: $\Longrightarrow$

Suppose $f_n \to f$ uniformly on E. Let $\varepsilon > 0$. Then $\exists N$ s.t.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \quad \forall x \in E, \ \forall n \geq N.$$

Now:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall x \in E, \ \forall n, m \geq N$$

$\Longleftarrow$: Suppose that Cauchy condition holds. Then, for each x, $\{f_n(x)\}$ is Cauchy and hence it converges to a limit, denoted as $f(x)$. Therefore,

$$f_n \to f \quad \text{pointwise on } E.$$

Let $\varepsilon > 0$. (CC) implies $\exists N$ s.t.:

$$|f_n(x) - f_m(x)| < \varepsilon, \quad x \in E, \quad n, m \geq N$$

Fix n and let $m \to \infty$:

$$\lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \varepsilon, \quad x \in E$$

$$|f_n(x) - \lim_{m \to \infty} f_m(x)| \leq \varepsilon \ ; \text{ since } t \mapsto |t| \text{ is continuou}$$

$$\Rightarrow \quad |f_n(x) - f(x)| \leq \varepsilon, \quad x \in E, \quad n \geq N$$

$\Rightarrow$ $f_n \to f$ uniformly on E.

Theorem 7.9 : Suppose:

$$\lim_{n\to\infty} f_n(x) = f(x),\ x\in E$$

Let $M_n = \sup_{x\in E} |f_n(x) - f(x)|$

Then:

$f_n \to f$ uniformly on $E \iff M_n \to 0$ as $n\to\infty$.

Proof:

$f_n\to f$ uniformly on $E \iff \forall \varepsilon>0,\ \exists N$ s.t $|f_n(x)-f(x)|<\varepsilon \quad \forall n\geq N,\ \forall x\in E$

$$\iff \sup_{x\in E} |f_n(x)-f(x)| \leq \varepsilon \quad \forall n\geq N$$

$$\iff M_n \leq \varepsilon,\ \forall n\geq N$$

$$\iff M_n \to 0 \text{ as } n\to\infty$$

Ex: Let $f_n(x) = \frac{x}{n}$, $x\in[0,1]$

Clearly, $f_n(x)\to 0$ pointwise.

$$\sup_{x\in[0,1]} |f_n(x) - 0| = \sup_{x\in[0,1]} \left|\frac{x}{n}\right| = \frac{1}{n} \to 0 \text{ as } n\to\infty.$$

Hence, theorem 7.9 implies that:

$f_n\to 0$ uniformly on $[0,1]$.

The corresponding theorem for series is:

<u>Theorem 7.10</u>: Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose that:

$$|f_n(x)| \leq M_n \ , \ x \in E, \quad n = 1, 2, 3, \dots$$

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on E if $\sum_{n=1}^{\infty} M_n$ converges.

<u>Proof</u>: If $\sum M_n$ converges, $\forall \varepsilon > 0$, $\exists N$ s.t:

$$\sum_{i=n}^{m} M_i < \varepsilon \ , \quad \forall m \geq n \geq N.$$

$$\Rightarrow \quad \left| \sum_{i=n}^{m} f_i(x) \right| \leq \sum_{i=n}^{m} |f_i(x)| \leq \sum_{i=n}^{m} M_i < \varepsilon, \ \forall x \in E$$

Therefore, the sequence of partial sums:

$$S_n(x) = \sum_{i=1}^{n} f_i(x)$$

satisfies:

$$|S_n(x) - S_m(x)| < \varepsilon, \quad \forall n, m \geq N, \ \forall x \in E$$

From Theorem 7.8 we conclude that $\{S_n(x)\}$ converges uniformly on E. ∎

<u>Ex</u> : Consider the series:

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad |x| \le 1$$

Since $\left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent,
Theorem 7.10 implies that $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges
uniformly on the interval $[-1, 1]$.