## Uniform convergence and continuity:

Theorem 7.11: Suppose $f_n \to f$ uniformly on $E$ in a metric space. Let $x$ be a limit point of $E$, and suppose that:

$$\lim_{t \to x} f_n(t) = A_n, \quad n = 1, 2, \dots$$

Then, $\{A_n\}$ converges and:

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

That is:

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

Proof: Let $\varepsilon > 0$.

Since $f_n \to f$ uniformly on $E$, $\exists N$ s.t

$$|f_n(t) - f_m(t)| < \varepsilon, \quad \forall n, m \geq N, \ \forall t \in E. \quad (1)$$

We let $t \to x$ in (1)

$$\lim_{t \to x} |f_n(t) - f_m(t)| \leq \varepsilon$$

$$\left| \lim_{t \to x} f_n(t) - \lim_{t \to x} f_m(t) \right| \leq \varepsilon$$

$$|A_n - A_m| \leq \varepsilon$$

$\Rightarrow \{A_n\}$ is Cauchy in $\mathbb{R}$ $\Rightarrow \exists A$ s.t. $A_n \to A$

• Then; $\exists N$ s.t:

$$|f(t)-A| \leq |f(t)-f_n(t)| + |f_n(t)-A_n| + |A_n - A|$$

$$\leq \underbrace{\frac{\varepsilon}{3}}_{\text{Since } f_n \to f \text{ unif}} + |f_n(t)-A_n| + \underbrace{\frac{\varepsilon}{3}}_{\text{Since } A_n \to A}, \quad \forall n \geq N$$

Since $\lim_{t\to x} f_N(t) = A_N$, $\exists N_r(x)$ s.t:

$$|f_N(t) - A_N| < \frac{\varepsilon}{3}, \quad \forall t \in N_r(x) \cap E, \ t \neq x.$$

$$\Rightarrow |f(t)-A| \leq \frac{\varepsilon}{3} + |f_N(t)-A_N| + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$$\Rightarrow |f(t)-A| < \varepsilon, \quad \forall t \in N_r(x) \cap E, \ t \neq x$$

Hence:

$$\lim_{t\to x} f(t) = A = \lim_{n\to\infty} A_n. \quad \blacksquare$$

<u>Theorem 7.12</u>: If $\{f_n\}$ is a sequence of continuous functions on $E$, and if $f_n \to f$ uniformly on $E$, then $f$ is continuous on $E$.

Proof :

Let $x \in E$

We need to show that:

$$\lim_{t \to x} f(t) = f(x)$$

Since $f_n$ is continuous on $E$, $n = 1, 2, \ldots$ :

$$\lim_{t \to x} f_n(t) = f_n(x).$$

Since $f_n(x) \to f(x)$ we have:

$$A_n := f_n(x) \longrightarrow A := f(x).$$

From Theorem 7.11 :

$$\lim_{t \to x} f(t) = A = \lim_{n \to \infty} A_n = f(x).$$

Hence:

$$\lim_{t \to x} f(t) = \lim_{t \to x} \lim_{n \to \infty} f_n(t) \quad ; \quad \text{because } f(t) = \lim_{n \to \infty} f_n(t)$$

$$= f(x)$$

$$= \lim_{n \to \infty} A_n$$

$$= \lim_{n \to \infty} f_n(x) \quad ; \quad \text{since } A_n = f_n(x)$$

$$= \lim_{n \to \infty} \lim_{t \to x} f_n(t) \quad ; \quad \text{because } f_n \text{ is cont.: } f_n(x) = \lim_{t \to x} f_n(t)$$

We conclude that $f$ is continuous on $E$ and :

$$\lim_{t \to \infty} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t). \qquad \blacksquare$$

<u>Remark</u>: The converse of Theorem 7.12 is not true. For example, let:

$$f_n(x) = \frac{x}{n}, \quad x \in \mathbb{R}$$

Clearly, $f_n$ is continuous, $n = 1, 2, \ldots$,

$$f_n \to 0 \text{ pointwise}$$

and $f(x) = 0$, $x \in \mathbb{R}$ is a continuous function. However, $f_n$ does not converge to $f$ uniformly. Indeed, we proceed by contradiction and assume that:

$$f_n \to 0 \text{ uniformly.}$$

Then, for $\varepsilon = \frac{1}{2}$, $\exists N$ s.t:

$$|f_n(x)| < \frac{1}{2}, \quad \forall n \geq N, \ \forall x \in \mathbb{R}.$$

But:

$$f_{N+1}(N+1) = \frac{N+1}{N+1} = 1 > \frac{1}{2}, \text{ which is a}$$

contradiction.

However, the following is true:

<u>Theorem 7.13</u>: Suppose $K$ is compact, and:

(a) $\{f_n\}$ is a sequence of continuous functions on $K$.

(b) $f_n \to f$ pointwise on $K$, $f$ is continuous.

(c) $f_n(x) \geq f_{n+1}(x)$, $\forall x \in K$, $n = 1, 2, \ldots$

Then, $f_n \to f$ uniformly on $K$

<u>Proof</u>: Let $g_n = f_n - f$.

We have:

$g_n$ is continuous, $g_n \to 0$ pointwise.

$g_n \geq g_{n+1}$, $g_n \geq 0$

We will prove that $g_n \to 0$ uniformly on $K$.

Let $\varepsilon > 0$.

Let: $K_n = \{x \in K : g_n(x) \geq \varepsilon\}$. $K_n$ is closed since $K_n = g_n^{-1}([\varepsilon, \infty))$ and $[\varepsilon, \infty)$ is closed in $\mathbb{R}$.

Then, since $K_n \subset K$ and $K$ is compact, we have that each $K_n$ is compact.

Also:

$$g_n \geq g_{n+1} \Rightarrow K_n \supset K_{n+1}$$

$$(\text{since } g_{n+1} \geq \varepsilon \Rightarrow g_n \geq \varepsilon).$$

Fix $x \in K$. Since $g_n(x) \to 0$, $\exists N(x)$ s.t:

$$|g_n(x)| < \frac{\varepsilon}{2}, \quad \forall n \geq N(x).$$

Hence:

$$x \notin K_n, \quad \forall n \geq N(x)$$

$$\Rightarrow x \notin \bigcap_{n=1}^{\infty} K_n.$$

Applying the argument to every $x \in K$ we conclude

$$\bigcap_{n=1}^{\infty} K_n = \emptyset \qquad (1)$$

Since $K_n \supset K_{n+1}$, $\exists N$ s.t. $K_N = \emptyset$, for otherwise, if $K_n \neq \emptyset$, $n = 1, 2, \cdots$, the Corollary of Theorem 2.36 implies that $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$, which contradicts (1).

Since $K_N = \emptyset$, then $K_n = \emptyset$, $\forall n \geq N$. Hence,

$$0 \leq g_n(x) < \varepsilon, \ \forall n \geq N, \ \forall x \in K.$$

$$\Rightarrow \ g_n \to 0 \text{ uniformly on } K$$

$$\Rightarrow \ f_n \to f \text{ uniformly on } K. \ \blacksquare$$

Ex: Let:

$$f_n(x) = \frac{x}{n}, \qquad x \in [0,1]$$

$f_n$ is continuous, $n = 1, 2, \ldots$

$f_n \to 0$ pointwise

$\frac{x}{n} \geq \frac{x}{n+1}$, $x \in [0,1] \Rightarrow f_n \geq f_{n+1}$

Theorem 7.13 gives;

$$f_n \to 0 \text{ uniformly on } [0,1].$$

<u>Def</u>: If $X$ is a metric space, $\mathcal{C}(X)$ is defined as:

$$\mathcal{C}(X) = \{ f : X \to \mathbb{R} : f \text{ is continuous and bounded on } X \}.$$

<u>Remark</u>: Boundednees in redundant in the definition of $\mathcal{C}(X)$ is $X$ is compact.

If $X$ is compact, $\mathcal{C}(X) = \{ f : X \to \mathbb{R} : f \text{ is continuous} \}$

If $f \in \mathcal{C}(X)$ we define:

$$\|f\| = \sup_{x \in X} |f(x)|$$

Clearly, $\|f\| < \infty$.

We have:

(1) $\|f\| = 0 \iff f(x) = 0 \quad \forall x \in X$

(2) If $h = f + g$ then, for any $x \in X$.

$$|h(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \underbrace{\|f\| + \|g\|}_{\text{upper bound.}}$$

$$\Rightarrow \sup_{x \in X} |h(x)| \leq \|f\| + \|g\|$$

$$\Rightarrow \|f + g\| \leq \|f\| + \|g\| \quad (*)$$

Define a distance in $\mathscr{C}(X)$ as follows:

$d(f,g) = \|f-g\|$

Lemma ; $d$ is a distance in $\mathscr{C}(X)$.

(a) $d(f,g) > 0$, $f \neq g$, $d(f,f) = \|f-f\| = 0$

(b) $d(f,g) = d(g,f)$

(c) $d(f,g) \leq d(f,h) + d(h,g)$.

Proof: (a) and (b) are clear. For (c):

$$d(f,g) = \|f-g\| = \|(f-h) + (h-g)\|;$$
$$\leq \|f-h\| + \|h-g\|; \text{ by } (*).$$

Therefore,

$$\boxed{(\mathscr{C}(X), d) \text{ is a metric space.}}$$

Recall Theorem 7.9:

- $f_n(x) \to f(x)$ pointwise, $x \in E$
- $M_n = \sup_{x \in E} |f_n(x) - f(x)|$. Then

$f_n \to f$ uniformly on $E \iff M_n \to 0$ $(**)$

We can rephrased Theorem 7.9 as follows;

$$\boxed{\text{Let } f_n, f \in \mathscr{C}(X). \text{ Then } f_n \to f \text{ uniformly on } X \text{ if and only if } \|f_n - f\| = d(f_n, f) \to 0 \text{ as } n \to \infty}$$

- <u>Theorem 7.15</u>. $(\mathcal{C}(X,d))$ is a complete metric space.

<u>Proof</u>: Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}(X)$.
Let $\varepsilon>0$. Then $\exists N$ s.t:

$$d(f_n, f_m)<\varepsilon, \ \forall n,m \geq N$$

That is:

$$\|f_n-f_m\| < \varepsilon, \ \forall n,m \geq N,$$

or:

$$|f_n(x)-f_m(x)| < \varepsilon \quad, n,m \geq N, \ \forall x\in X.$$

From Theorem 7.8, $\exists f: X\to \mathbb{R}$ such that:

$$f_n \to f \text{ uniformly on } X.$$

Since $f_n$, $n=1,2,\dots$ is continuous, from Theorem 7.12, $f$ is also continuous.
We need to check that <u>$f\in \mathcal{C}(X)$</u> and $d(f_n,f)\to 0$.

(a) $f$ is bounded, since for $\varepsilon=1$, $\exists N$ s.t:

$$|f(x)-f_n(x)|<1, \ \forall x\in X, \ n\geq N$$

$$\begin{aligned}
\Rightarrow |f(x)| &= |f(x)-f_N(x)+f_N(x)| \\
&\leq |f(x)-f_N(x)| + |f_N(x)| \\
&\leq 1 + |f_N(x)| \\
&\leq 1+M_N \ ; \text{ since } f_N \text{ is bounded}
\end{aligned}$$

$\Rightarrow$ $f$ is bounded and continuous $\Rightarrow$ <u>$f\in \mathcal{C}(X)$</u>.

(b) $f_n\to f$ uniformly is equivalent to $\|f_n-f\|\to 0$.
That is, $d(f_n,f)\to 0$. Since $f_n\to f$ in $\mathcal{C}(X)$ we conclude that $\mathcal{C}(X)$ is complete. ∎