Uniform Convergence and integration.

<u>Theorem 7.16</u>: Let $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$, $n = 1, 2, \ldots$ Suppose that $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and

$$\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha.$$

<u>Proof</u>: Let

$$\varepsilon_n = \sup_{a \le x \le b} |f_n(x) - f(x)|.$$

From Theorem 7.9, $\varepsilon_n \to 0$

We have:

$$|f_n(x) - f(x)| \le \varepsilon_n, \quad \forall x \in [a,b]$$

$$\Rightarrow \quad -\varepsilon_n \le f(x) - f_n(x) \le \varepsilon_n, \quad x \in [a,b]$$

$$\Rightarrow \quad f_n(x) - \varepsilon_n \le f(x) \le \varepsilon_n + f_n(x), \quad x \in [a,b]$$

<u>Note</u>: $g \le h \Rightarrow L(P, g, \alpha) \le L(P, h, \alpha)$ and $U(P, g, \alpha) \le U(P, h, \alpha)$

Then:

$$L(P, f_n - \varepsilon_n, \alpha) \le L(P, f, \alpha)$$

$$\Rightarrow \underline{\int_a^b} (f_n - \varepsilon_n)\, d\alpha \le \underline{\int_a^b} f\, d\alpha$$

$$\Rightarrow \int_a^b (f_n - \varepsilon_n)\, d\alpha = \underline{\int_a^b} f\, d\alpha; \quad \text{because } f_n - \varepsilon_n \in \mathcal{R}(\alpha) \qquad (1)$$

Similarly:

$$U(P, f, \alpha) \leq U(P, f_n + \varepsilon_n, \alpha)$$

$$\Rightarrow \overline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} (f_n + \varepsilon_n) \, d\alpha$$

$$\Rightarrow \overline{\int_a^b} f \, d\alpha \leq \int_a^b (f_n + \varepsilon_n) \, d\alpha; \quad \text{because } f_n + \varepsilon_n \in \mathcal{R}(\alpha) \qquad (2)$$

From (1) and (2):

$$\int_a^b (f_n - \varepsilon_n) \, d\alpha \leq \underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha \leq \int_a^b (f_n + \varepsilon_n) \, d\alpha$$

Thus:

$$0 \leq \overline{\int_a^b} f \, d\alpha - \underline{\int_a^b} f \, d\alpha \leq \int_a^b 2\varepsilon_n \, d\alpha$$

$$0 \leq \overline{\int_a^b} f \, d\alpha - \underline{\int_a^b} f \, d\alpha \leq 2\varepsilon_n \left(\alpha(b) - \alpha(a)\right)$$

Letting $\varepsilon_n \to 0$, the squeeze theorem yields:

$$\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha,$$

which means that $f \in \mathcal{R}(\alpha)$. We now compute:

$$0 \leq \left| \int_a^b f_n \, d\alpha - \int_a^b f \, d\alpha \right| \leq \int_a^b |f_n(x) - f(x)| \, d\alpha$$

$$\leq \varepsilon_n \int_a^b d\alpha = \varepsilon_n \left(\alpha(b) - \alpha(a)\right)$$

$$\Rightarrow \quad 0 \leq \left| \int_a^b f_n \, d\alpha - \int_a^b f \, d\alpha \right| \leq \varepsilon_n \left(\alpha(b) - \alpha(a)\right).$$

We let $\varepsilon_n \to 0$ and use the squeeze theorem again to conclude:

$$\lim_{n\to\infty} \left| \int_a^b f_n \, d\alpha - \int_a^b f \, d\alpha \right| = 0,$$

that is:

$$\lim_{n\to\infty} \int_a^b f_n \, d\alpha = \int_a^b f \, d\alpha. \blacksquare$$

<u>Corollary</u>: If $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and if:

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad a \le x \le b$$

where the series is converging uniformly on $[a,b]$. Then:

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$$

That is, we can integrate the series term by term.

<u>Proof</u>:

$$s_n(x) = \sum_{i=1}^{n} f_n(x)$$

We have that $s_n(x) \to f(x)$ uniformly. Then:

$$\int_a^b f \, d\alpha = \int_a^b \lim_{n\to\infty} s_n \, d\alpha$$

$$= \lim_{n\to\infty} \int_a^b s_n \, d\alpha; \text{ from Theorem 7.16}$$

$$= \lim_{n\to\infty} \int_a^b \sum_{i=1}^{n} f_i \, d\alpha$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \int_a^b f_i \, d\alpha$$

$$= \sum_{i=1}^{\infty} \int_a^b f_i \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha.$$

Uniform Convergence and Differentiation

<u>Theorem 7.17</u>: Suppose $\{f_n\}$ is a sequence of functions differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If:

$\{f_n'\}$ converges uniformly on $[a,b]$,

then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and:

$$f'(x) = \lim_{n\to\infty} f_n'(x), \quad a \le x \le b$$

<u>Proof</u>: Let $\varepsilon > 0$. Since $\{f_n(x_0)\}$ converges and $\{f_n'\}$ converges uniformly on $[a,b]$, $\exists N$ s.t

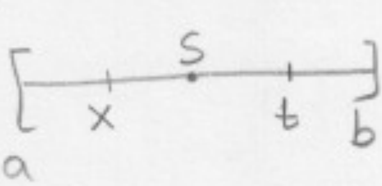
$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}, \quad \forall n,m \geq N. \qquad (1)$$

and

$$|f_n'(x) - f_m'(x)| < \frac{\varepsilon}{2(b-a)}, \quad \forall n,m \geq N, \ \forall x \in [a,b]. \qquad (2)$$

We apply the mean value theorem to $f_n - f_m$:

$$|(f_n - f_m)(x) - (f_n - f_m)(t)| = |(f_n' - f_m')(s)|\,|t-x|$$



$$< \frac{\varepsilon}{2(b-a)}|t-x| \ ; \text{ from } (2)$$

$$\leq \frac{\varepsilon}{2\cancel{(b-a)}}\cancel{(b-a)} \ ; \text{ since } |t-x| \leq b-a$$

$$= \frac{\varepsilon}{2}$$

• We have:

$$(3) \quad |(f_n - f_m)(x) - (f_n - f_m)(t)| < \frac{\varepsilon}{2}, \quad \forall n, m \geq N, \ \forall x, t, \ a \leq x < t \leq b$$

We compute:

$$|f_n(x) - f_m(x)| = |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0) + f_n(x_0) - f_m(x_0)|$$

$$\leq |(f_n - f_m)(x) - (f_n - f_m)(x_0)| + |f_n(x_0) - f_m(x_0)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ ; \quad \text{by (1) and (3)}$$

$$= \varepsilon$$

$$\Rightarrow \quad |f_n(x) - f_m(x)| < \varepsilon, \ \forall n, m \geq N, \ \forall x \in [a, b].$$

$\Rightarrow$ $\{f_n\}$ converges uniformly on $[a, b]$. Let:

$$f(x) := \lim_{n \to \infty} f_n(x), \quad x \in [a, b].$$

Fix $x \in [a, b]$ and define:

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \quad \phi(t) = \frac{f(t) - f(x)}{t - x},$$

for $t \in (a, b)$, $t \neq x$.

We have, since f_n is differentiable:

$$\lim_{t \to x} \phi_n(t) = f_n'(x), \quad n = 1, 2, 3, \ldots$$

- We compute:

$$|\phi_n(t) - \phi_m(t)| = \left| \frac{f_n(t)-f_n(x)}{t-x} - \frac{f_m(t)-f_m(x)}{t-x} \right|$$

$$= \left| \frac{f_n(t) - f_m(t) - (f_n(x) - f_m(x))}{t-x} \right|$$

$$= \left| \frac{(f_n - f_m)(t) - (f_n - f_m)(x)}{t-x} \right|$$

$$< \frac{\varepsilon}{2(b-a)} \frac{|t-x|}{|t-x|} \quad ; \quad \text{by (2) and the mean value theorem}$$

$$= \frac{\varepsilon}{2(b-a)}, \quad \forall n,m \geq N, \ \forall t \neq x$$

This implies:

$\{\phi_n\}$ converges uniformly on $[a,b] \setminus \{x\}$

Since $f_n \to f$ we have:

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \frac{f_n(t) - f_n(x)}{t-x}$$

$$= \frac{f(t) - f(t)}{t-x} = \phi(t)$$

We have shown:

$$(*)\begin{cases} \phi_n \to \phi \text{ uniformly on } [a,b]\setminus\{x\} \\ \text{and} \\ \lim\limits_{t\to x} \phi_n(t) = f_n'(x) \end{cases}$$

With $A_n := f_n'(x)$, from $(*)$ and Theorem 7.11 we obtain:

$$\lim_{n\to\infty} f_n'(x) = \lim_{t\to x} \phi(t),$$

that is, f is differentiable at x, and:

$$\boxed{\lim_{n\to\infty} f_n'(x) = f'(x) \quad a \le x \le b.}$$

∎

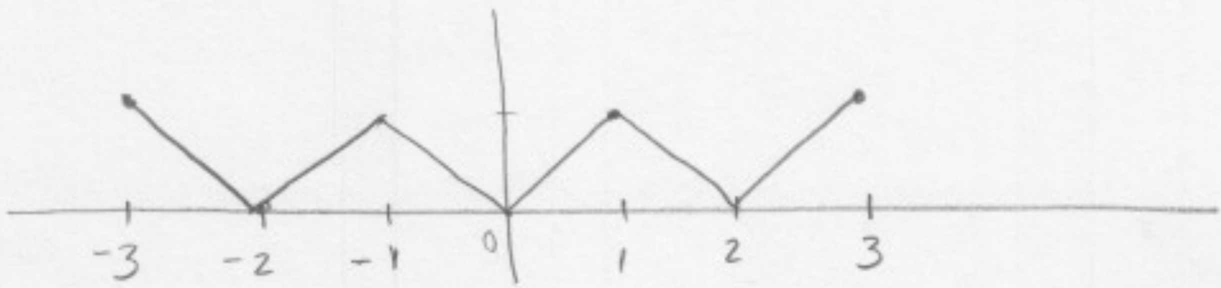
Theorem 7.18. There exists a real continuous function on the real line which is nowhere differentiable.

Proof: Consider

$$\varphi(x) = |x|, \quad -1 \leq x \leq 1$$

Extend $\varphi(x)$ to $\mathbb{R}$ as follows:

$$\varphi(x+2) = \varphi(x)$$



φ is continuous on $\mathbb{R}$

Define:

$$f_n(x) = \left(\frac{3}{4}\right)^n \varphi(4^n x)$$

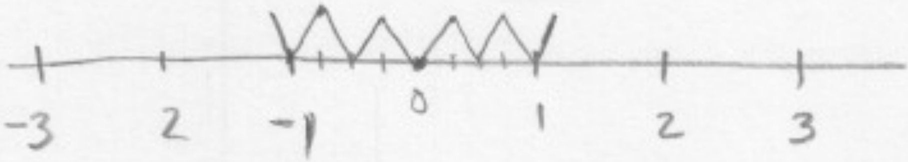
We have:

$$|f_n(x)| \leq \left(\frac{3}{4}\right)^n, \quad \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n < \infty$$

Theorem 7.10 gives:

$\sum_{n=0}^{\infty} f_n$ converges uniformly on $\mathbb{R}$ to a continuous function, say f:

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$



$f_1(x) = \frac{3}{4}\varphi(4x)$

Theorem 7.12 $\Rightarrow$ f is continuous on $\mathbb{R}$.

Fix a real number x and a positive integer m.
Let:

$$\delta_m = \pm \frac{1}{2} 4^{-m}$$

where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. This can be done since $4^m |\delta_m| = \frac{1}{2}$. We estimate:

$$\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m} \right|$$

Let $\gamma_n := \dfrac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}$

<u>Case 1</u>: $n > m$. Then $4^n \delta_m = \pm \dfrac{4^n}{2 \cdot 4^m} = \pm \dfrac{4^{n-m}}{2}$

Hence, $\varphi(4^n x + 4^n \delta_m) = \varphi(4^n x)$ and

$\gamma_n = 0$

Case 2 : If $0 \le n \le m$ then, since

$$|\varphi(s) - \varphi(s)| \le |s-t|$$
$$\shortparallel$$
$$|\,|s| - |t|\,|$$

we compute :

$$|\gamma_n| = |\varphi(4^n x + 4^n \delta_m) - \varphi(4^n x)|$$
$$\le 4^n |\delta_m| \le 4^n,$$

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n\right| ; \quad \text{since } \gamma_n = 0 \text{ for } n > m.$$

$$= \left|\sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \gamma_n + \left(\frac{3}{4}\right)^m \gamma_m\right|$$

$$\ge \left(\frac{3}{4}\right)^m |\gamma_m| - \left|\sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \gamma_n\right|$$

$$= \left(\frac{3}{4}\right)^m 4^m - \left|\sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \gamma_n\right| ; \quad \text{because } |\gamma_m| = 4^m$$

$$\ge 3^m - \sum_{n=0}^{m-1} 3^n ; \quad \text{because } \left|\sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \gamma_n\right| \le \sum_{n=0}^{m-1} 3^n$$

$$= 3^m - \frac{1-3^m}{1-3} = 3^m + \frac{1}{2} - \frac{3^m}{2}$$

$$= \frac{3^m}{2} + \frac{1}{2} = \frac{1+3^m}{2}$$

As $m \to \infty$, $\delta_m \to 0$ and clearly, f can not be differentiable at x because $\frac{(1+3^m)}{2} \to \infty$. ▫