Equicontinuous families of functions

Recall Theorem 3.6:

(a) If $\{p_n\}$ is a sequence in a compact metric space X, then there exists a subsequence $\{p_{n_k}\}$ that converges to a point of X.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

We want a similar theorem for functions.

Definition: Let $f_n : E \to \mathbb{R}$. We say that $\{f_n\}$ is pointwise bounded on E if $\exists\, \phi : E \to \mathbb{R}$ such that:

$$|f_n(x)| < \phi(x), \quad x \in E, \quad \forall n = 1, 2, 3, \dots$$

$\{f_n\}$ is uniformly bounded on E if $\exists M$ s.t.:

$$|f_n(x)| < M \quad \forall x \in E, \ \forall n = 1, 2, 3, \dots$$

Ex: This example will show that $\{f_n\}$ being uniformly bounded does not imply that there exists a subsequence $\{f_{n_k}\}$ that converges pointwise

Let $f_n(x) = \sin nx, \quad x \in [0, 2\pi]$

Clearly, $|f_n(x)| \leq 1 \ \forall x \in [0, 2\pi], \ \forall n = 1, 2, \dots$. Hence, $\{f_n\}$ is uniformly bounded. We proceed by

- contradiction and assume that there exists a subsequence $\{f_{n_k}\} = \{\sin n_k x\}$ such that, for each $x \in [0,2]$:

$$\{\sin n_k x\} \text{ is a convergent sequence. } (*)$$

From $(*)$, $\{\sin n_k x\}$ is Cauchy and hence:

$$\lim_{k\to\infty} (\sin n_k x - \sin n_{k+1} x) = 0, \quad 0 \le x \le 2\pi$$

$$\Rightarrow$$

$$(1) \quad \lim_{k\to\infty} (\sin n_k x - \sin n_{k+1} x)^2 = 0, \quad 0 \le x \le 2\pi.$$

Because we only have pointwise convergence in (1) we can not use Theorem 7.16 (which requires uniform convergence). However, we will use a more general theorem (Theorem 11.32, chapter 11) which is the Lebesgue Dominated Convergence Theorem:

$$\boxed{0 = \int_0^{2\pi} 0 \, dx = \lim_{k\to\infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 \, dx} \quad (1)$$

But:

$$\int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 dx$$

$$= \int_0^{2\pi} (\sin^2 n_k x - 2 \sin n_k x \sin n_{k+1} x + \sin^2 n_{k+1} x)\, dx$$

<u>Recall</u>: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta = 1 - \cos 2\theta \quad \Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \cos^2\theta - (1-\cos^2\theta)$$
$$= 2\cos^2\theta - 1$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$\Rightarrow$

$$\int_0^{2\pi} \sin^2 n_k x \, dx = \int_0^{2\pi}\left(\frac{1}{2} - \frac{\cos 2n_k x}{2}\right)dx$$
$$= \frac{1}{2}(2\pi) - \frac{1}{4n_k}\Big[\sin 2n_k x\Big]_0^{2\pi}; \text{ by fundamental theorem of Calculus.}$$
$$= \pi - 0$$

$$\int_0^{2\pi} \sin^2 n_{k+1} x \, dx = \pi$$

We also have:

$$\int \sin ax \sin bx \, dx = \frac{\sin (a-b)x}{2(a-b)} - \frac{\sin (a+b)x}{2(a+b} + C, \quad a^2 \neq b^2$$

$$\Rightarrow \int_0^{2\pi} \sin n_k x \sin n_{k+1} x \, dx = 0$$

$$\Rightarrow \int_0^{2\pi}\left(\sin^2 n_k x - 2\sin n_k x \sin n_{k+1} x + \sin^2 n_{k+1} x\right)dx = 2\pi$$

which contradicts (1), since $2\pi \neq 0$.

We conclude that $\{\sin nx\}$ does not have a pointwise convergent subsequence.

Ex : This example will show that $\{f_n\}$ being pointwise convergent does not imply that there exists a subsequence $\{f_{n_k}\}$ that converges uniformly.

Let $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2}$, $x \in [0,1]$, $n = 1, 2, 3, \ldots$

$|f_n(x)| \le 1$, $\forall x \in [0,1]$, $n = 1, 2, \ldots$

$\Rightarrow \{f_n\}$ is uniformly bounded on $[0,1]$.

Also:

$$\lim_{n\to\infty} f_n(x) = 0, \quad 0 \le x \le 1$$

We proceed by contradiction and assume that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges uniformly to 0; i.e.,

$$f_{n_k} \to 0 \text{ uniformly}$$

Then, for $\varepsilon = \frac{1}{2}$, $\exists N$ s.t.:

$$|f_{n_k}(x)| < \tfrac{1}{2}, \quad \forall x \in [0,1], \ \forall k \ge N.$$

But, for $k \ge N$ we compute:

$$f_{n_k}\left(\frac{1}{n_k}\right) = \frac{x^2}{x^2 + \left(1 - n_k\left(\frac{1}{n_k}\right)\right)^2} = 1 > \frac{1}{2},$$

which is a contradiction. We conclude that $\{f_n\}$ does not have a uniformly convergent subsequence.

We need to find the right conditions on a sequence $\{f_n\}$ to guarantee the existence of a convergent subsequence.

<u>Definition</u>: A family $\mathcal{F}$ of functions $f: E \to \mathbb{R}$, $E \subset X$, X a metric space, is said to be equicontinuous on E if:

$\forall \varepsilon > 0$, $\exists \delta > 0$, such that:

$d(x,y) < \delta$, $x, y \in E$, $\Rightarrow |f(x) - f(y)| < \varepsilon$, $\forall f \in \mathcal{F}$.

<u>Ex</u>: The sequence $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2}$, $x \in [0,1]$ is not equicontinuous. In order to see this we proceed by contradiction. Then, if $\{f_n\}$ is equicontinuous, for $\varepsilon = \frac{1}{2}$, $\exists \delta > 0$ s.t.

$$|x-y| < \delta, \; x, y \in [0,1] \Rightarrow |f_n(x) - f_n(y)| < \tfrac{1}{2}, \; \forall n = 1, 2, \ldots$$

Choose N so that $\frac{1}{N} < \delta$. Then:

$$\left|\tfrac{1}{N} - 0\right| < \delta \quad \text{but} \quad \left|f_N\left(\tfrac{1}{N}\right) - f_N(0)\right| = |1 - 0| = 1 > \tfrac{1}{2},$$

which is a contradiction. We conclude that $\{f_n\}$ is not equicontinuous. ∎

- <u>Ex</u>: Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of differentiable functions such that $\{f_n'\}$ is uniformly bounded. Then $\{f_n\}$ is equicontinuous. Indeed, we have:

$$|f_n'(x)| \le M, \quad \forall x \in [a,b]. \qquad (1)$$

We apply the mean value theorem to each f_n. For every $a \le x < y \le b$:

$$f_n(x) - f_n(y) = f_n'(c)(x-y), \quad \text{for some } c \in (x,y).$$

c depends on n, but condition (1) yields:

$$\begin{aligned} |f_n(x) - f_n(y)| &= |f_n'(c)|\,|x-y| \\ &\le M\,|x-y| \end{aligned}$$

$$\Rightarrow |f_n(x) - f_n(y)| \le M\,|x-y|, \quad \forall x,y \in [a,b], \ \forall n = 1,2,3,\dots$$

Let $\varepsilon > 0$.
Define $\delta = \frac{\varepsilon}{M}$, then:

$$\text{if } |x-y| < \delta,\ x,y \in [a,b] \Rightarrow \begin{aligned}[t] |f_n(x) - f_n(y)| &\le M|x-y| \\ &\le M\delta \\ &= M\frac{\varepsilon}{M} = \varepsilon \end{aligned}$$

$\Rightarrow \{f_n\}$ is equicontinuous.

<u>Remark</u>: Every member of an equicontinuous family is uniformly continuous.

<u>Theorem 7.23</u>: If $\{f_n\}$ is a pointwise bounded sequence of functions, $f_n: E \to \mathbb{R}$, where E is a countable set, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that

$$\{f_{n_k}(x)\} \text{ converges for every } x \in E.$$

<u>Proof</u>:

Let $E = \{x_1, x_2, \cdots\}$.

Since $\{f_n(x_1)\}$ is a bounded sequence of real numbers (because $\{f_n\}$ is pointwise bounded), Theorem 3.6 implies that $\{f_n(x_1)\}$ has a convergent subsequence. Then, there exists a subsequence of $\{f_n\}$, denoted as $\{f_{1,k}\}$ such that

$$\{f_{1,k}(x_1)\} \text{ converges as } k \to \infty.$$

Since $\{f_{1,k}(x_2)\}$ is bounded, the same reasoning implies that there exists a subsequence of $\{f_{1,k}\}$, denoted as $\{f_{2,k}\}$ such that:

$$\{f_{2,k}(x_2)\} \text{ converges as } k \to \infty.$$

We continue in this way. We have:

Subsequence 1: $f_{1,1}$ $f_{1,2}$ $f_{1,3}$ $f_{1,4}$ ....

Subsequence 2: $f_{2,1}$ $f_{2,2}$ $f_{2,3}$ $f_{2,4}$ ....

Subsequence 3: $f_{3,1}$ $f_{3,2}$ $f_{3,3}$ $f_{3,4}$ ....

$\vdots$

(a) $\{f_{n,k}\}_{k=1}^{\infty}$ is a subsequence of $\{f_{n-1,k}\}_{k=1}^{\infty}$ for $n = 2, 3, 4, \ldots$

(b) $\{f_{n,k}(x_n)\}$ converges as $k \to \infty$.

(c) The order in which the functions appear is the same in each sequence.

We now take the diagonal of the array:

$$f_{1,1} \quad f_{2,2} \quad f_{3,3} \quad f_{4,4} \quad \cdots$$

The sequence $\{f_{n,n}\}_{n=1}^{\infty}$ (except possibly its first $n-1$ terms) is a subsequence of $\{f_{n,k}\}_{k=1}^{\infty}$, $n = 1, 2, 3 \ldots$. Therefore:

$\{f_{n,n}(x_i)\}$ converges as $n \to \infty$, for every $x_i \in E$. ∎