Recall the definition of a dense set.

<u>Def</u>: E is a dense subset of the metric space X if every point of X is a limit point of E, or a point of E (or both). That is,

For every $p \in X$, $\exists \{p_n\}$, $p_n \in E$ s.t.:

$$d(p_n, p) \to 0$$

<u>Def</u>: A metric space is called <u>separable</u> if it contains a countable dense subset.

<u>Ex</u>: $\mathbb{R}^k$ is separable, since:

$$\mathbb{Q}^k = \{(p_1, \dots, p_k) : p_i \text{ is rational}\}$$

is a countable dense subset of $\mathbb{R}^k$.

<u>Def</u>: A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X if for every $x \in X$ and every open set $G \subset X$ such that $x \in G$ we have that:

$$\exists V_\alpha \text{ s.t. } x \in V_\alpha \subset G, \text{ for some } \alpha.$$

In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Ex: Solve Problem 23, chapter 2:

Show that every separable metric space X has a countable basis.

Proof: X has a countable dense subset E.

$$E = \{x_1, x_2, \dots\}$$

Consider the collection of all neighborhoods $\{N_q(x_i)\}$

$$N_q(x_i) = \{x : d(x, x_i) < q\}, \quad q \in \mathbb{Q}$$

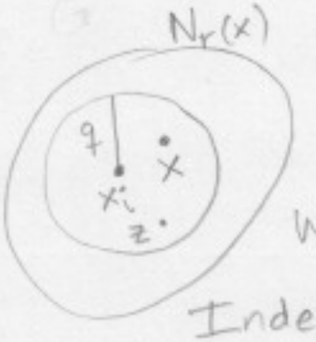
The collection $\{N_q(x_i)\}$ is countable since $\mathbb{Q}$ is countable and it is a base for X. Indeed:

If $G \subset X$ is open and $x \in G$, $\exists r > 0$ s.t:

$$N_r(x) \subset G$$; because x is an interior point

Since E is dense in X, $\exists x_i \in E$ s.t:

$$d(x, x_i) < \frac{r}{4}$$



Let $q \in \mathbb{Q}$ s.t. $\frac{r}{4} < q < \frac{r}{2}$

We claim that $\underline{x \in N_q(x_i) \subset N_r(x) \subset G}$:

Indeed:

$$d(x, x_i) < \frac{r}{4} < q \implies x \in N_q(x_i)$$

Let $z \in N_q(x_i) \implies d(z, x) \le d(z, x_i) + d(x_i, x)$

$$< q + \frac{r}{4}$$

$$< \frac{r}{2} + \frac{r}{2} = r$$

$$\implies d(z, x) < r \implies z \in N_r(x). \quad \blacksquare$$

Ex: Solve problem 25, chapter 2:
Prove that every compact metric space K has a countable base, and that K is separable.

Proof: For each $n = 1, 2, \cdots$, we have:

$$K = \bigcup_{p \in K} N_{\frac{1}{n}}(p), \text{ open cover of } K.$$

Since K is compact $\Rightarrow \exists p_1^n, p_2^n, \dots, p_{r_n}^n \in K$ s.t:

$$K \subset N_{\frac{1}{n}}(p_1^n) \cup \dots \cup N_{\frac{1}{n}}(p_{r_n}^n) \quad (*)$$

We will now show that the countable collection:

$$\{N_{\frac{1}{n}}(p_i^n)\} \quad n = 1, 2, \dots \quad i = 1, 2, \dots, r_n$$

is a basis of E.

Let G be an open set and $x \in G$. $\Rightarrow \exists r > 0$ s.t. $N_r(x) \subset G$.

Fix n such that $\frac{1}{n} < \frac{r}{2}$.

Since $x \in K$, $(*)$ implies that $x \in N_{\frac{1}{n}}(p_i^n)$ for some $i \in \{1, \dots, r_n\}$

Also, $N_{\frac{1}{n}}(p_i^n) \subset N_r(x)$ because $z \in N_{\frac{1}{n}}(p_i^n) \Rightarrow$

$d(z, x) \leq d(z, p_i^n) + d(p_i^n, x) < \frac{1}{n} + \frac{1}{n} < \frac{r}{2} + \frac{r}{2} = r$

Hence $z \in N_r(x)$. We have proved:

$$x \in N_{\frac{1}{n}}(p_i^n) \subset G.$$

To show that K is separable, define $E = \bigcup_{n=1}^{\infty} A_n$, $A_n = \{p_1^n, \dots, p_{r_n}^n\}$. E is a countable dense subset of K.

Theorem 7.25: Let K be a compact metric space. If $f_n \in \mathscr{C}(K)$, $n = 1, 2, \dots$ and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then:

(a) $\{f_n\}$ is uniformly bounded on K

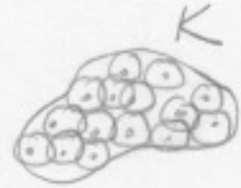
(b) $\{f_n\}$ contains a uniformly convergent subsequence.

(This is Arzela-Ascoli Theorem).

Proof: Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t:

$$d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon, \ \forall n \qquad (1)$$

We have:



$$K \subset \bigcup_{p \in K} N_\delta(p), \text{ open cover of } K.$$

Since K is compact, $\exists p_1, \dots, p_r \in K$ s.t:

$$K \subset N_\delta(p_1) \cup \dots \cup N_\delta(p_r) \qquad (2)$$

Since $\{f_n\}$ is pointwise bounded $\Rightarrow \exists M_i$ s.t:

$$|f_n(p_i)| < M_i \quad \forall n$$

If $M = \max\{M_1, \dots, M_r\}$, then for every $x \in K$, $x \in N_\delta(p_i)$ for some $i \in \{1, \dots, r\}$ and:

$$|f_n(x) - f_n(p_i)| < \varepsilon, \quad n = 1, 2, \dots$$

$$\Rightarrow |f_n(x)| \le |f_n(x) - f_n(p_i)| + |f_n(p_i)| < \varepsilon + M_i \le \varepsilon + M.$$

$$\Rightarrow |f_n(x)| < \varepsilon + M \quad \forall x \in K, \forall n.$$

$\Rightarrow \{f_n\}$ is uniformly bounded on K.

We showed earlier that any compact metric space contains a countable dense subset.

Let E be a countable dense subset of K.

From Theorem 7.23 it follows that $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ such that:

$f_{n_i}(y)$ converges for every $y \in E$.

We define:

$$\boxed{g_i = f_{n_i}}$$

We will prove that f_{n_i} (i.e., g_i) converges uniformly on K.

Let $\varepsilon > 0$ and $\delta > 0$ as in **(1)**.

Let $E = \{y_1, y_2, y_3, \dots\}$. Clearly:

$$K \subset \bigcup_{i=1}^{\infty} N_\delta(y_i), \text{ open cover of } K$$

(since $x \in K \Rightarrow \exists y_i$ s.t $d(x, y_i) < \delta \Rightarrow x \in N_\delta(y_i)$)

K compact $\Rightarrow \exists x_1, x_2, \dots, x_m \in E$ s.t;

$$K \subset N_\delta(x_1) \cup \dots \cup N_\delta(x_m)$$

Since $\{g_i(y)\}$ converges for every $y \in E$, $\exists N$ s.t:

$$\boxed{(**) \quad |g_i(x_s) - g_j(x_s)| < \varepsilon, \quad \forall i, j \geq N, \quad 1 \leq s \leq m}$$

Note that $\{x_1, \dots, x_m\}$ is a subset of $\{y_i : i = 1, 2, \dots\}$

- Let $x \in K$.

$\Rightarrow \exists s \in \{1, \dots, m\}$ s.t $x \in N_\delta(x_s)$.

We compute:

$$|g_i(x) - g_j(x)| \leq |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)|$$

$$< \underbrace{\varepsilon}_{\text{by } (1)} + \underbrace{\varepsilon}_{\text{by } (**)} + \underbrace{\varepsilon}_{\text{by } (1)}$$

We have proved:

$$|f_{n_i}(x) - f_{n_j}(x)| < 3\varepsilon \quad , \forall x \in K, \quad i,j \geq N$$

which implies that:

$\{g_i\} = \{f_{n_i}\}$ converges uniformly on K. ∎