The Stone - Weierstrass Theorem

Theorem 7.26 : If $f : [a,b] \to \mathbb{R}$ is a continuous function on $[a,b]$, then there exists a sequence of polynomials P_n such that:

$$\lim_{n \to \infty} P_n(x) = f(x),$$

uniformly on $[a,b]$.

Proof :
Without loss of generality we assume $[a,b] = [0,1]$

Case 1 : $f(0) = f(1) = 0$; $f : [0,1] \to \mathbb{R}$ continuous

We extend the function f to the real line as $f(x) = 0$, $x \notin [0,1]$. Clearly, f is uniformly continuous on the whole line.

We define for $-1 \leq x \leq 1$ the function:

$$\boxed{Q_n(x) = c_n (1 - x^2)^n, \quad n = 1, 2, \ldots \quad (1)}$$

where c_n is chosen so that:

$$\int_{-1}^{1} Q_n(x)\, dx = 1$$

We will show that:

$$c_n < \sqrt{n}, \quad n = 1, 2, \ldots \quad (\ast\ast\ast)$$

• In order to obtain (*) we integrate (1):

$$\int_{-1}^{1}(1-x^2)^n dx = \int_{-1}^{1}\frac{Q_n(x)}{c_n}dx = \frac{1}{c_n}\int_{-1}^{1}Q_n(x) = \frac{1}{c_n}$$

But $\int_{-1}^{1}(1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx$,

and hence:

$$\frac{1}{c_n} = 2\int_0^1 (1-x^2)^n dx \geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx \quad .(2)$$

We claim that $(1-x^2)^n \geq 1-nx^2$. Indeed, for $x \in [0,1]$ we define $g(x) = (1-x^2)^n - 1 + nx^2$. Note that $g(0)=0$

We compute:

$$g'(x) = n(1-x^2)^{n-1}(-2x) + 2nx$$
$$= -2nx(1-x^2)^{n-1} + 2nx > 0 \quad (\text{since } x \in (0,1)).$$

$g'(x) > 0$ implies that $g(x)$ is increasing on $(0,1)$, and, since $g(0)=0$ we obtain that $g(x) \geq 0$ on $(0,1)$. That is $(1-x^2)^n \geq 1-nx^2$, $x \in (0,1)$.

From (2):

$$\frac{1}{c_n} \geq 2\int_0^{\frac{1}{\sqrt{n}}}(1-nx^2)dx$$

$$= \frac{2}{\sqrt{n}} - 2n\left[\frac{x^3}{3}\right]_0^{\frac{1}{\sqrt{n}}}, \text{ by fundamental theorem of calculus}$$

$$= \frac{2}{\sqrt{n}} - \frac{2n}{3}\frac{1}{n^{3/2}}$$

Hence:

$$\frac{1}{c_n} \geq \frac{2}{\sqrt{n}} - \frac{2}{3\sqrt{n}} = \frac{6-2}{3\sqrt{n}} = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}} .$$

$$\Rightarrow \quad \frac{1}{c_n} > \frac{1}{\sqrt{n}} , \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad c_n < \sqrt{n} , \quad n = 1, 2, \ldots, \text{ which is } (***)$$

We define the sequence of polynomials $\{P_n\}$ as follows:

$$\boxed{P_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt , \qquad 0 \leq x \leq 1}$$

$$P_n(x) = \int_{-x}^{1-x} f(x+t)\, Q_n(t)\, dt ;$$

because $f \equiv 0$ outside $[0,1]$ we have:
$x + t \geq 1 \Leftrightarrow t \geq 1 - x$
$x + t \leq 0 \Leftrightarrow t \geq -x$

$$\therefore \quad P_n(x) = \int_{0}^{1} f(z)\, Q_n(z-x)\, dz ;$$

because of Theorem 6.19 (change of Variable):
$z = x + t$, $t = z - x$, $dz = dt$
$t = -x \Rightarrow \underline{z = 0}$; $t = 1 - x \Rightarrow \underline{z = 1}$

Clearly, P_n is a polynomial in x:

$$P_n(x) = c_n \int_0^1 f(t)\, (1 - (t-x)^2)^n\, dt$$
$$= c_n \int_0^1 f(t)\, (1 - t^2 + 2tx - x^2)^n\, dt$$
$$= A_1 + A_2 x + A_3 x^2 + \cdots + A_{2n+1} x^{2n}$$

$\{P_n\}$ is a sequence of polynomials.

- Let $\varepsilon > 0$

Since f is absolutely continuous on $\mathbb{R}$,

$$\boxed{\begin{array}{l} \exists\, \delta > 0 \ \text{ s.t:} \\ |x-y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2} \quad (3) \end{array}}$$

Let

$$M := \sup |f(x)|$$

For $0 \le x \le 1$ we estimate;

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} f(x+t) Q_n(t)\, dt - f(x) \int_{-1}^{1} Q_n(t)\, dt \right|$$

$$= \left| \int_{-1}^{1} \left[f(x+t) - f(x) \right] Q_n(t)\, dt \right|$$

$$\le \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

$$= \underbrace{\int_{-1}^{-\delta} |f(x+t) - f(x)| Q_n(t)\, dt}_{(A)} + \underbrace{\int_{-\delta}^{\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt}_{(B)}$$

$$+ \underbrace{\int_{\delta}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt}_{(C)}$$

We need to show that each term (A), (B) and (C) is small.

For **(A)**, since $c_n < \sqrt{n}$ and $\sup |f(x)| \le M$;

$$\int_{-1}^{-\delta} |f(x+t) - f(x)| \, Q_n(t)\, dt \le \int_{-1}^{-\delta} \big(|f(x+t)| + |f(x)| \big) Q_n(t)\, dt$$

$$\le 2M \int_{-1}^{-\delta} \sqrt{n}\, (1-\delta^2)^n \, dt, \text{ by note below}$$

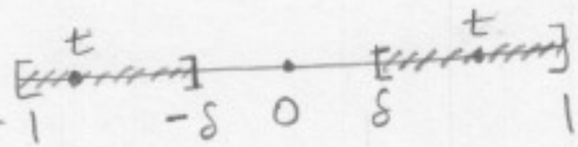
$$= 2M \sqrt{n}\, (1-\delta^2)^n \big(-\delta - (-1)\big)$$

For **(C)**, since $C_n < \sqrt{n}$, in the same way we obtain:

$$\int_{\delta}^{1} |f(x+t) - f(x)| \, Q_n(t)\, dt \le 2M \sqrt{n}\, (1-\delta^2)^n (1-\delta)$$

$\boxed{\text{Note}}$: $Q_n(t) \le \sqrt{n}\, (1-\delta^2)^n$ because if ;

$\delta \le |t| \le 1$



then $t^2 \ge \delta^2 \Rightarrow -t^2 \le -\delta^2 \Rightarrow 1 - t^2 \le 1 - \delta^2$

and $Q_n(t) = C_n (1-t^2)^n \le \sqrt{n}\, (1-\delta^2)^n$.

For (B) we use (3) to obtain:

$$\int_{-\delta}^{\delta} | f(x+t) - f(x) | \, Q_n(t)\, dt < \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\, dt$$

$$\le \frac{\varepsilon}{2} \int_{-1}^{1} Q_n(t)\, dt$$

$$= \frac{\varepsilon}{2} \; ; \quad \text{since } \int_{-1}^{1} Q_n(t)\, dt = 1$$

We have shown that;

$$(A)+(B)+(C) < 4M\sqrt{n}(1-\delta^2)^n(1-\delta)+\frac{\varepsilon}{2}$$

that is; $< 4M\sqrt{n}(1-\delta^2)^n+\frac{\varepsilon}{2}$; since $0<1-\delta<1$.

$$\boxed{|P_n(x)-f(x)| < 4M\sqrt{n}(1-\delta^2)^n+\frac{\varepsilon}{2}, \quad 0\le x\le 1 \quad (4)}$$

We saw earlier that:

$Q_n(x) \le \sqrt{n}(1-\delta^2)^n$ if $\delta \le |x| \le 1$. We will show now that $\sqrt{n}(1-\delta^2)^n \to 0$ as $n\to\infty$. In particular this implies that $Q_n \to 0$ uniformly on $\delta \le |x| \le 1$.

Let $a_n = \sqrt{n}(1-\delta^2)^n$ and $b_n = a_n^{1/n}$.

$$\Rightarrow \quad b_n = n^{\frac{1}{2n}}(1-\delta^2)$$

$$= (n^{1/n})^{1/2}(1-\delta^2)$$

In a previous lecture we proved: $\sqrt[n]{n} \to 1$ as $n\to\infty$.

$\therefore \lim\limits_{n\to\infty} b_n = (1-\delta^2)$. Let $\eta>0$ s.t. $\eta+(1-\delta^2)<1$

Since $\lim\limits_{n\to\infty} a_n^{1/n} = (1-\delta^2)$, for $\eta>0$, $\exists N$ s.t;

$$|a_n^{1/n}-(1-\delta^2)| < \eta, \quad \forall n \ge N$$

$\therefore \quad a_n^{1/n} < \eta+(1-\delta^2)$; let $\alpha = \eta+(1-\delta^2)$, we have $\alpha<1$

Therefore:

$$a_n^{1/n} < \alpha, \quad \forall n \geq N$$

$$\therefore \quad a_n < \alpha^n, \quad \forall n \geq N$$

$$\therefore \quad 0 < a_n < \alpha^n \quad \forall n \geq N.$$

Letting $n \to \infty$, the squeeze theorem yields, since $\alpha^n \to 0$:

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

That is:

$$\boxed{\lim_{n\to\infty} \sqrt{n}\,(1-\delta^2)^n = 0}$$

Going back to (4), $a_n \to 0$ gives that $\exists N$ such that:

$$\sqrt{n}\,(1-\delta^2)^n < \frac{\varepsilon}{8M}, \quad \forall n \geq N.$$

Then:

$$|P_n(x) - f(x)| < 4M\sqrt{n}\,(1-\delta^2)^n + \frac{\varepsilon}{2}$$
$$< 4M \cdot \frac{\varepsilon}{8M} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \geq N$$

$$\Rightarrow \boxed{\begin{array}{l} |P_n(x) - f(x)| < \varepsilon, \quad \forall x \in [0,1], \ \forall n \geq N \\ \Rightarrow \ P_n \to f \text{ uniformly on } [0,1]. \end{array}}$$

- We still need to consider:

<u>Case 2</u>; $f(0) \neq 0$ or $f(1) \neq 0$, $f:[0,1] \to \mathbb{R}$.

In this case we define;

$g(x) = f(x) - f(0) - x\,[f(1) - f(0)]$, $0 \leq x \leq 1$

Clearly, $g(0) = 0$, $g(1) = 0$, so $g:[0,1] \to \mathbb{R}$ falls in case 1. Then, $\exists \{P_n\}$ a sequence of polynomials on $[0,1]$ s.t;

$$P_n \to g \text{ uniformly on } [0,1].$$

Define:

$$q_n = P_n + f(0) + x\,[f(1) - f(0)]$$

$\{q_n\}$ is a sequence of polynomials and:

$$q_n(x) \longrightarrow g(x) + f(0) + x\,[f(1) - f(0)] \text{ uniformly.}$$

The definition of $g(x)$ yields;

$$q_n(x) \to f(x) \text{ uniformly on } [0,1]. \quad \blacksquare$$