We will now prove a generalization of the Stone-Weierstrass theorem called the Stone's theorem. For this we need the following Corollary of the Stone-Weierstrass theorem:

Corollary: For every interval $[-a, a]$ there is a sequence of polynomials $\{P_n\}$ such that $P_n(0)=0$ and such that:

$$\lim_{n\to\infty} P_n(x) = |x|,$$

uniformly on $[-a, a]$.

Proof: By Theorem 7.26, there exists a sequence $\{P_n^*\}$ of polynomials which converges to $|x|$ uniformly on $[-a, a]$. So

$$P_n^* \to |x| \text{ uniformly on } [-a, a].$$

Since $x \mapsto |x|$ has value 0 at $x=0$, and $P_n^*(0) \to |0|$ as $n \to \infty$ we can define the sequence of polynomials:

$$P_n(x) = P_n^*(x) - P_n^*(0), \quad n = 1, 2, \dots$$

Clearly, $P_n(0) = P_n^*(0) - P_n^*(0) = 0$ as desired and, since $P_n^*(0) \to 0$ as $n \to \infty$ we have:

$$P_n \to |x| \text{ uniformly on } [-a, a].$$

Recall that we have defined the space of functions:

$$\mathscr{C}([a,b]) = \{ f : [a,b] \to \mathbb{R} : f \text{ is continuous} \},$$

and that $\mathscr{C}([a,b])$ is a metric space with the distance function:

$$d(f,g) = \|f-g\|,$$

where the norm $\|\cdot\|$ is defined as;

$$\|h\| = \sup_{x \in [a,b]} |h(x)|, \quad h \in \mathscr{C}([a,b]).$$

Recall Theorem 2.27;
Let X be a metric space and $E \subset X$. Then:
(a) $\bar{E}$ is closed
(b) $E = \bar{E} \iff E$ is closed.

Recall that $\bar{E} = E \cup E'$, where E' is the set of limit points of E.

- We have the following:

<u>Corollary</u>: Consider the metric space $X = (\mathscr{C}([a,b]), d)$. Let $\mathscr{P}([a,b])$ be the set of all polynomials $p : [a,b] \to \mathbb{R}$. Then:

$$\overline{\mathscr{P}([a,b])} = \mathscr{C}([a,b]) = X$$

<u>Proof</u>: If $p \in \mathscr{P}([a,b])$ then $p \in \mathscr{C}([a,b])$ since all polynomials are continuous. Therefore:

$$\mathscr{P}([a,b]) \subset X$$

Let f be a limit point of $\mathscr{P}([a,b])$. Then there exists a sequence $\{p_n\}$, $p_n \in \mathscr{P}([a,b])$ such that:

$p_n \to f$ in X; that is, $d(p_n, f) \to 0$.

we remarked earlier that:

$d(p_n, f) \to 0 \iff \|p_n - f\| \to 0 \iff p_n \to f$ uniformly.

Hence, since $\{p_n\}$ is a sequence of continuous functions converging uniformly to f, Theorem 7.12 implies that f is continuous. That is, $f \in X$. Thus, with $E := \mathscr{P}([a,b])$, we have shown that:

$$E \cup E' \subset X \implies \overline{E} \subset X$$

$$\implies \overline{\mathscr{P}([a,b])} \subset \mathscr{C}([a,b]). \qquad (1)$$

- Conversely, let $f \in X$.

The Stone-Weierstrass theorem gives the existence of a sequence of polynomials $\{p_n\}$ on $[a,b]$ such that:

$$p_n \to f \text{ uniformly on } [a,b].$$

We use again the fact that the following three statements are equivalent:

$$p_n \to f \text{ uniformly} \iff \|p_n - f\| \to 0 \iff d(p_n, f) \to 0.$$

Now, if $f \in \mathcal{P}([a,b])$, then clearly $f \in \overline{\mathcal{P}([a,b])}$, since the closure of $\mathcal{P}([a,b])$ is the union of the set $\mathcal{P}([a,b])$ with $\mathcal{P}([a,b])'$, the set of all limit points of $\mathcal{P}([a,b])$. Assume now that $f \notin \mathcal{P}([a,b])$. Since $d(p_n, f) \to 0$, for every $\delta > 0$, the neighborhood:

$$N_\delta(f) = \{h \in X : d(h,f) < \delta\},$$

contains a polynomial p_n (actually an infinite number of polynomials from $\{p_n\}$). This shows that f is a limit point of $\mathcal{P}([a,b])$, and hence $f \in \overline{\mathcal{P}([a,b])}$. We have shown that:

$$X \subset \overline{\mathcal{P}([a,b])} \qquad (2)$$

From (1) and (2) we conclude:

$$\overline{\mathcal{P}([a,b])} = \mathcal{C}([a,b]). \quad \blacksquare$$

<u>Remark</u> : Previous Corollary shows that $\mathcal{P}([a,b])$, the set of all polynomials on $[a,b]$, is a dense subset of the metric space $\mathcal{C}([a,b])$.