# Algebras of functions.

Def: A family $\mathcal{A}$ of real-valued functions $f: E \to \mathbb{R}$ is said to be an algebra if:

(i) $f, g \in \mathcal{A} \Rightarrow f+g \in \mathcal{A}$

(ii) $f, g \in \mathcal{A} \Rightarrow fg \in \mathcal{A}$

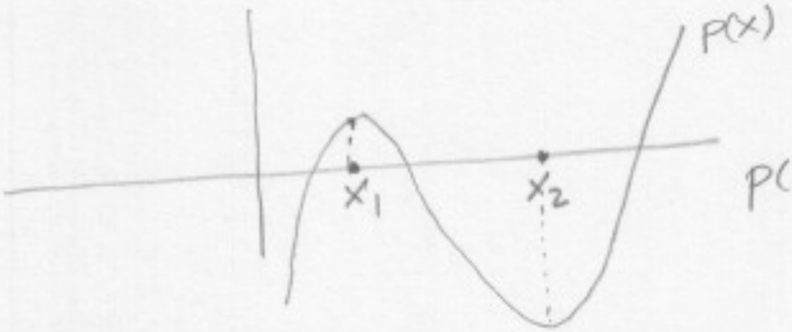
(iii) $f \in \mathcal{A},\ c \in \mathbb{R} \Rightarrow cf \in \mathcal{A}$.

Ex : The set of all polynomials is an algebra.

Definition : Let $\mathcal{A}$ be a family of functions on a set $E$. Then $\mathcal{A}$ is said to separate points on $E$ if for every pair of distincts points $x_1, x_2 \in E$, there exists a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.

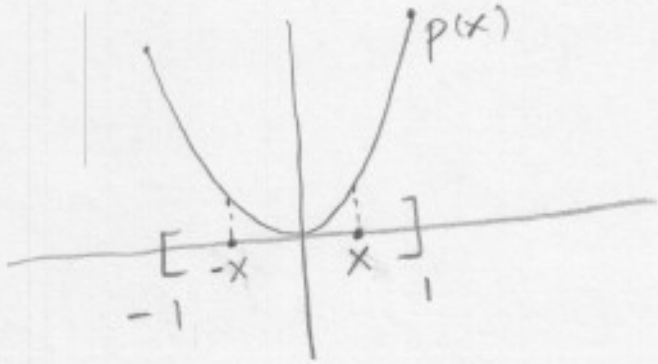
Definition : Let $\mathcal{A}$ be a family of functions on a set $E$. We say that $\mathcal{A}$ vanishes at no point of $E$ if for every $x \in E$, there exists a function $g \in \mathcal{A}$ such that $g(x) \neq 0$.

Ex: The algebra of all polynomials in one variable separates points on $\mathbb{R}$.




$p(x_1) \neq p(x_2)$.

Ex: The algebra of all even polynomials on $[-1,1]$ does not separates points since $p(-x) = p(x)$ for every even polynomial $p$.



Theorem 7.31: Suppose $\mathcal{A}$ is an algebra of functions on a set $E$, assume also that:

(a) $\mathcal{A}$ separates points on $E$

(b) $\mathcal{A}$ vanishes at no point on $E$

Suppose $x_1, x_2$ are distincts points of $E$, and $c_1, c_2$ are constants. Then $\mathcal{A}$ contains a function $f$ such that:

$$f(x_1) = c_1, \quad f(x_2) = c_2.$$

Proof: Hypothesis (a) and (b) imply that there exists $g, h, K \in \mathcal{A}$ such that:

$$g(x_1) \neq g(x_2), \quad h(x_1) \neq 0, \quad K(x_2) \neq 0. \qquad (*)$$

Define:

$$u = gK - g(x_1)K, \quad v = gh - g(x_2)h$$

Since $\mathcal{A}$ is an algebra we have that $u, v \in \mathcal{A}$

Clearly,

$$u(x_1) = g(x_1)K(x_1) - g(x_1)K(x_1) = 0$$

$$v(x_2) = g(x_2)h(x_2) - g(x_2)h(x_2) = 0$$

$$u(x_2) = g(x_2)K(x_2) - g(x_1)K(x_2) \neq 0, \text{ by } (*)$$

$$v(x_1) = g(x_1)h(x_1) - g(x_2)h(x_1) \neq 0, \text{ by } (*)$$

Therefore, we can define:

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

f is the desired function because:

$$f(x_1) = \frac{c_1 v(x_1)}{v(x_1)} + \cancel{\frac{c_2 \, u(x_1)}{u(x_2)}}^{\,0} = c_1$$

$$f(x_2) = \cancel{\frac{c_1 v(x_2)}{v(x_1)}}^{\,0} + \frac{c_2 \, u(x_2)}{u(x_2)} = c_2 .$$

We will use the following:

<u>Lemma</u>: Let $\mathcal{A}$ be an algebra of continuous functions on a compact set $K$ of a metric space. That is:

$$\mathcal{A} \subset \mathscr{C}(K),$$

$$\mathscr{C}(K) = \{ f : K \to \mathbb{R} : f \text{ is continuous} \}.$$

Then $\bar{\mathcal{A}}$, the closure of $\mathcal{A}$, is also an algebra of continuous functions, $\bar{\mathcal{A}} \subset \mathscr{C}(K)$.

<u>Proof</u>: Let $f, g \in \bar{\mathcal{A}}$. Then $\exists \{f_n\}$, $\exists \{g_n\}$ such that $d(f_n, f) \to 0$ and $d(g_n, g) \to 0$, which is equivalent to:

$$f_n \to f \text{ uniformly on } K$$

$$g_n \to g \text{ uniformly on } K$$

Theorem 7.12 $\Rightarrow$ $f, g$ are continuous on $K$.

$$\Rightarrow f, g \in \mathscr{C}(K)$$

Exercise #2 in Chapter 7 gives that:

$\{f_n + g_n\}$ converges uniformly on $K$

$\{f_n g_n\}$ converges uniformly on $K$

$cf_n$ converges uniformly on $K$, $c \in \mathbb{R}$.

Clearly,

$$f_n + g_n \to f + g, \quad f_n g_n \to fg \quad \text{and} \quad cf_n \to f,$$

uniformly on $K$.

That is

$$d(f_n + g_n, f+g) \to 0$$
$$d(f_n g_n, fg) \to 0$$
$$d(cf_n, cf) \to 0,$$

which means $f+g$, $fg$ and $cf$ are limit points of $A$, and hence, since $\bar{A} = A \cup A'$ we conclude:

$$f+g \in \bar{A}, \quad fg \in \bar{A}, \quad cf \in \bar{A}.$$

So the closed set $\bar{A} \subset \mathscr{C}(K)$ is also an algebra of continuous functions defined on $K$. ∎

We can now state:

Theorem 7.32 (Stone's generalization of the Stone-Weierstrass theorem).

Let $\mathcal{A}$ be an algebra of real continuous functions on a compact set $K$. That is:

$$\mathcal{A} \subset \mathscr{C}(K),$$

$$\mathscr{C}(K) = \{ f : K \to \mathbb{R} : f \text{ is continuous} \}$$

If $\mathcal{A}$ separates points on $K$ and if $\mathcal{A}$ vanishes at no point of $K$, then

$$\overline{\mathcal{A}} = \mathscr{C}(K),$$

that is, $\mathcal{A}$ is dense in $\mathscr{C}(K)$