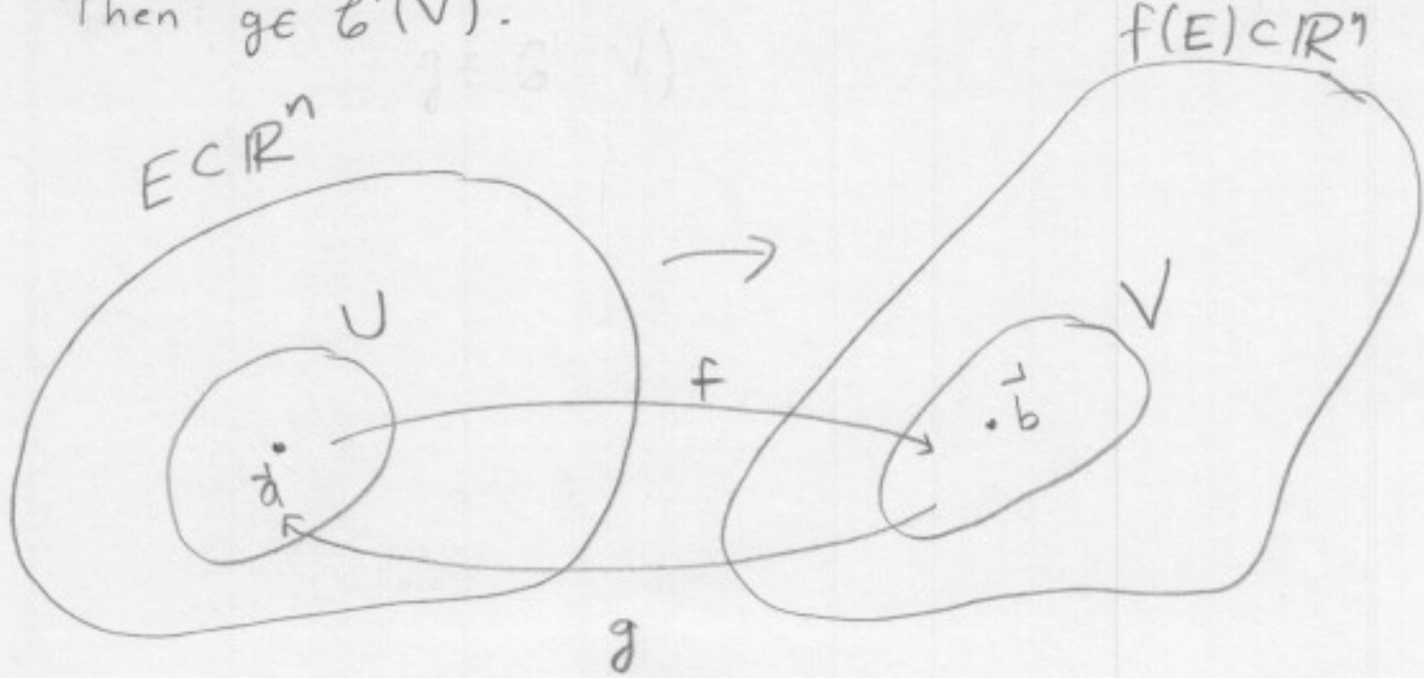
The inverse function Theorem and the implicit function theorem.

Theorem 9.24. The inverse function Theorem.
Suppose $f: E \subset \mathbb{R}^n \to \mathbb{R}^n$ is a map of class $\mathscr{C}^1$ (i.e. f is differentiable at each point $\vec{x} \in E$, and the function $\vec{x} \mapsto f'(\vec{x})$ is a continuous function).

Suppose that $f'(\vec{a})$ is invertible for some $\vec{a} \in E$, anb $\vec{b} = f(\vec{a})$. Then;

(a) There exist open sets U and V in $\mathbb{R}^n$ such that $\vec{a} \in U$, $\vec{b} \in V$, $f(U) = V$ and f is 1-1 on U.

(b) Let $g = f^{-1} : V \longrightarrow U$ (i.e., $g(f(\vec{x})) = \vec{x}$, $\forall \vec{x} \in U$) Then $g \in \mathscr{C}^1(V)$.



Ex. Consider the equations:

$$u = \frac{x^4+y^4}{x}, \quad v = \sin x + \cos y.$$

Near which points (x,y) can we solve for x, y in terms of u, v?

We have:

$$f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \quad f \in \mathcal{C}^1(\mathbb{R}^2 \setminus \{0\})$$

$$f(x,y) = \left(\frac{x^4+y^4}{x}, \sin x + \cos y \right)$$

$$f_1(x,y) = \frac{x^4+y^4}{x}, \quad f_2(x,y) = \sin x + \cos y.$$

$$f'(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{3x^4-y^4}{x^2} & \frac{4y^3}{x} \\ \cos x & -\sin y \end{pmatrix}$$

$f'(x,y)$ is a linear map from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}^2$

$f'(x,y) \in L(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}^2)$.

$f'(x,y)$ is invertible $\iff$ det (matrix) $\neq 0$

$\iff \frac{\sin y}{x^2}(y^4 - 3x^4) - \frac{4y^3}{x}\cos x$ is not zero

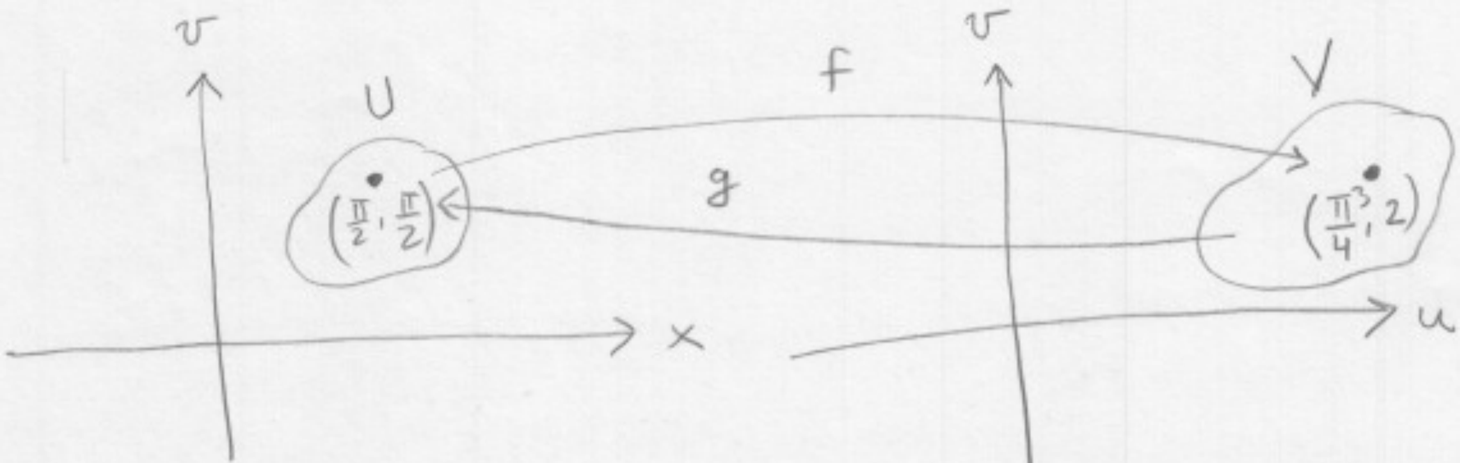
- Therefore, we can solve x, y:

$$x = x(u,v)$$
$$y = y(u,v)$$

near all points (x,y) for which:

$$x \neq 0 \quad \text{and} \quad (\sin y)(y^4 - 3x^4) \neq 4xy^3 \cos x$$

For example, if $x_0 = \frac{\pi}{2}$, $y_0 = \frac{\pi}{2}$ we can solve for x, y near (x_0, y_0) because the determinant is not zero.



$$f\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) = \left(\frac{\frac{\pi^4}{16} + \frac{\pi^4}{16}}{\frac{\pi}{2}}, 2 \right) = \left(\frac{\pi^3}{4}, 2 \right)$$

$f: U \to V$ is 1-1 and on-to

The inverse $g: V \to U$ is also continuously differentiable, i.e., $g \in \mathcal{C}^1(V)$.

The following theorem will be used in the proof of the inverse function theorem.

Def 9.22 ; Let X be a metric space, with metric d. If $\varphi : X \to X$ satisfies that:

$$d(\varphi(x), \varphi(y)) \le c\, d(x,y) \quad \forall x, y \in X, \qquad (1)$$

where $c<1$, then φ is said to be a contraction of X into X.

Theorem 9.23 : If X is a complete metric space, and if $\varphi : X \to X$ is a contraction, then there exists one and only one $x \in X$ such that $\varphi(x) = x$.

Proof :

Uniqueness : If $\varphi(x) = x$ and $\varphi(y) = y$, then

since
$$d(\underset{\overset{\shortparallel}{x}}{\varphi(x)}, \underset{\overset{\shortparallel}{y}}{\varphi(y)}) \le c\, d(x,y) \quad , \qquad c<1$$

$$\Rightarrow \quad d(x,y) \le c\, d(x,y),$$

which can only happen when $d(x,y) = 0$.

Existence : Pick $x_0 \in X$ arbitrarily, define:

$$x_{n+1} = \varphi(x_n), \quad n = 0, 1, 2, \ldots$$

For some $c<1$ we have:

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1}))$$
$$\leq c\, d(x_n, x_{n-1})$$

Hence induction gives:

$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0), \quad n = 0, 1, 2, \ldots$$

If $n < m$, it follows that:

$$d(x_n, x_m) \leq \sum_{i=n+1}^{m} d(x_i, x_{i-1})$$
$$\leq (c^n + c^{n+1} + \ldots + c^{m-1})\, d(x_1, x_0)$$
$$\leq c^n(1 + c + c^2 + \ldots)\, d(x_1, x_0)$$
$$= c^n \cdot \frac{1}{1-c}\, d(x_1, x_0)$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is comple, $\lim x_n = x$, for some $x \in X$.

Since φ is a contraction, φ is continuous (actually φ is uniformly continuous). Hence:

$x_n \to x$ implies that $\varphi(x_n) \to \varphi(x)$

$$\therefore \quad \varphi(x) = \lim_{n\to\infty} \varphi(x_n)$$
$$= \lim_{n\to\infty} x_{n+1}; \quad \text{since } x_{n+1} = \varphi(x_n)$$
$$= x$$

$$\Rightarrow \quad \varphi(x) = x.$$

Proof of the inverse function Theorem:

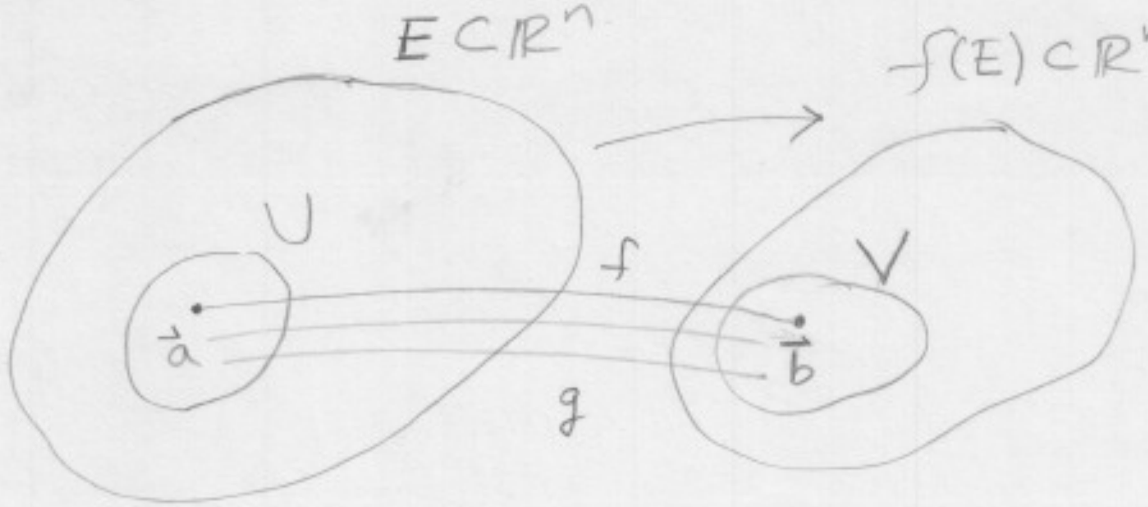
<u>Thm</u>: Suppose $f: E \subset \mathbb{R}^n \to \mathbb{R}^n$ is a map of class $\mathscr{C}'$ (i.e. f is differentiable at each point $x \in E$, and $x \mapsto f'(x)$ is a continuous function).

Suppose that $\vec{a} \in U$ is such that $f'(\vec{a})$ is invertible. Then

(a) $\exists\, U, V$ in $\mathbb{R}^n$, $\vec{a} \in U$, $\vec{b} \in V$, $f(\vec{a}) = \vec{b}$ such that $f(U) = V$ and f is 1-1 on U

(b) Let $g = f^{-1}: V \to U$ (i.e. $g(f(x)) = x$, $x \in U$). Then:

$$g \in \mathscr{C}'(V).$$



Proof:

Interpretation:

$$y = f(x)$$

$$(y_1, \dots, y_n) = f(x_1, \dots, x_n) = \big(f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)\big)$$

$$\left.\begin{aligned} y_1 &= f_1(x_1, \dots, x_n) \\ y_2 &= f_2(x_1, \dots, x_n) \\ &\vdots \\ y_n &= f_n(x_1, \dots, x_n) \end{aligned}\right\} n \text{ equations.}$$

Equations can be solved in small neighborhoods of a and b.

$$\left.\begin{aligned} x_1 &= x_1(y_1, \dots, y_n) \\ &\vdots \\ x_n &= x_n(y_1, \dots, y_n). \end{aligned}\right\} \text{continuously differentiable functions.}$$

Proof:

- Let $A = f'(a)$, choose λ so that

$$2\lambda \|A^{-1}\| = 1$$

- $f \in \mathscr{C}'$ $\Rightarrow$ f' is continuous on E. In particular f' is continuous at $x = a$. Hence, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\|f'(x) - \underset{A}{\underset{\shortparallel}{f'(a)}}\| < \varepsilon \quad \text{if} \quad |x - a| < \delta.$$

In particular, for $\varepsilon = \lambda$, $\exists \delta_0$ s.t:

$$\boxed{x \in B_{\delta_0}(a) := U \implies \|f'(x) - A\| < \lambda.} \quad (1)$$

- Fix $y \in \mathbb{R}^n$. We define

$$\varphi(x) = x + A^{-1}(y - f(x)), \quad x \in E$$

Note: $f(x) = y \iff \varphi(x) = x$

$$\begin{aligned}
\varphi'(x) &= I - A^{-1}f'(x) \\
&= A^{-1}A - A^{-1}f'(x) \\
&= A^{-1}(A - f'(x)) \\
\|\varphi'(x)\| &= \|A^{-1}(A - f'(x)\| \\
&\le \|A^{-1}\| \, \|A - f'(x)\| \\
&< \|A^{-1}\| \cdot \lambda, \quad \underline{\text{if } x \in U} \\
&= \frac{1}{2}
\end{aligned}$$

$$\Rightarrow \boxed{|\varphi(x_1) - \varphi(x_2)| \le \frac{1}{2}|x_1 - x_2|, \quad x_1, x_2 \in U}$$

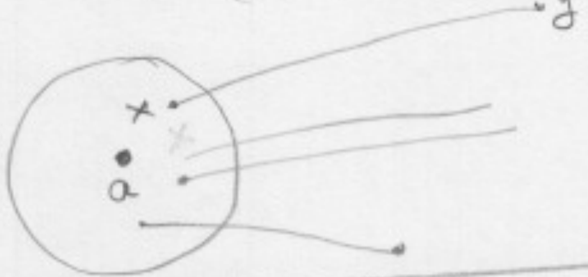
Recall: $\gamma(t) = (1-t)x_1 + tx_2 \qquad 0 \le t \le 1$

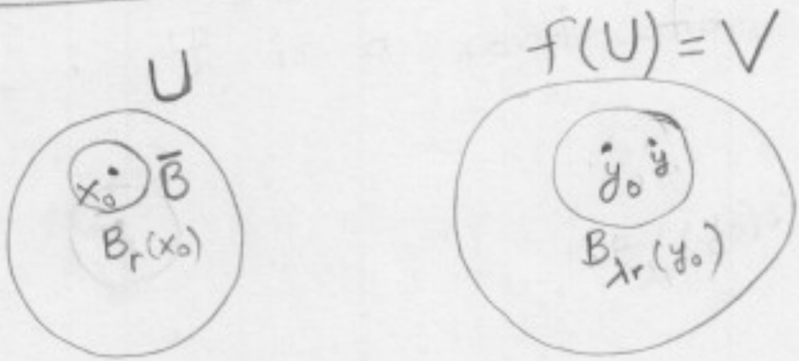
$$\begin{aligned}
h(t) &= \varphi(\gamma(t)) \\
h'(t) &= \varphi'(\gamma(t)) \cdot \gamma'(t) \\
h'(t) &= \varphi'(\gamma(t)) \cdot (X_2 - X_1) \\
\int_0^1 h'(t) &= h(1) - h(0) \\
&= \varphi(\gamma(1)) - \varphi(\gamma(0)) \\
&= \varphi(X_2) - \varphi(X_1)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow |\varphi(x_2) - \varphi(x_1)| &= \left|\int_0^1 h'(t)\,dt\right| \\
&\le \int_0^1 \|h'(t)\|\,dt \\
&= \int_0^1 \|\varphi'(\gamma(t))(x_2 - x_1)\|\,dt \\
&\le \int_0^1 \|\varphi'(\gamma(t))\| \, |x_2 - x_1|\,dt \\
&< \frac{1}{2}|x_2 - x_1|
\end{aligned}$$

- $|\varphi(x_1)-\varphi(x_2)| \le \frac{1}{2}|x_1-x_2| \quad \forall x_1,x_2 \in U$ implies that φ has at most one fixed point. If $\varphi(x_1)=x_1$ and $\varphi(x_2)=x_2$ $\Rightarrow |x_1-x_2| \le \frac{1}{2}|x_1-x_2| \Rightarrow x_1=x_2$.

- Hence $\varphi(x)=x+A^{-1}(y-f(x))$, $x\in U$ implies that $y=f(x)$ for at most one $x\in U$; that is, f is 1-1.





$f(x_0)=y_0$

Fix $y\in B_{\lambda r}(y_0)$

Let $B=B_r(x_0)$, $\overline{B}\subset U$.

Claim: $B_{\lambda r}(y_0)\subset V$; i.e. V is open

$$
\begin{aligned}
x\in\overline{B} \Rightarrow |\varphi(x)-x_0| &\le |\varphi(x)-\varphi(x_0)| + |\varphi(x_0)-x_0| \\
&\le \tfrac{1}{2}|x-x_0| + |A^{-1}(y-f(x_0))| \\
&\le \tfrac{1}{2}|x-x_0| + \|A^{-1}\|\,|y-y_0| \\
&\le \tfrac{1}{2}|x-x_0| + \|A^{-1}\|\,\lambda r \\
&= \tfrac{1}{2}|x-x_0| + \tfrac{r}{2} \le r \Rightarrow \varphi(x)\in\overline{B}
\end{aligned}
$$

Hence:

$$\begin{cases} \varphi : \overline{B} \to \overline{B} \\ |\varphi(x_1) - \varphi(x_2)| \le \frac{1}{2}|x_1 - x_2|, \ \forall x_1, x_2 \in \overline{B} \end{cases}$$

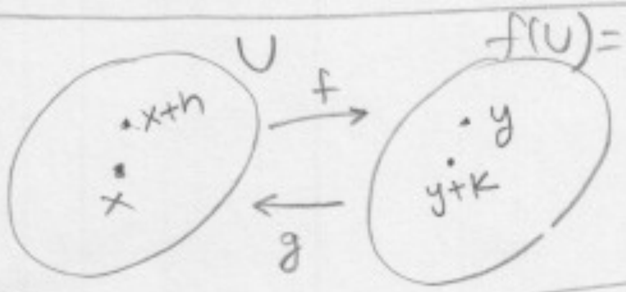
Contraction $\Rightarrow$ $\exists$ unique fixed point

$x \in \overline{B}$, $\varphi(x) = x$ and hence $\underline{f(x) = y}$.

Hence $y \in f(\overline{B}) \subset f(U) = V \Rightarrow B_{\lambda r}(y_0) \subset V$

$\Rightarrow V$ is open.

(b)



$y + K = f(x+h)$ some K.

$y = f(x)$

Thm 9.8: Let $\Omega = \{L : \mathbb{R}^n \to \mathbb{R}^n$, linear invertible$\}$

(a) If $A \subset \Omega$, $B : \mathbb{R}^n \to \mathbb{R}^n$ linear and

If $\|B - A\| \, \|A^{-1}\| < 1$, then $B \in \Omega$.

$$\varphi(x+h) = x + h + A^{-1}(y - f(x+h))$$

$$\varphi(x) = x + A^{-1}(y - f(x))$$

$$\varphi(x+h) - \varphi(x) = h + A^{-1}(f(x) - f(x+h))$$

$$= h - A^{-1}K$$

$$|\varphi(x+h) - \varphi(x)| = |h - A^{-1}K| \le \frac{1}{2}|h|$$

$$|h| - |A^{-1}K| \le |h - A^{-1}K| \le \frac{1}{2}|h|$$

$$\Rightarrow |A^{-1}K| \ge \frac{1}{2}|h|$$

$$\Rightarrow |h| \le 2\|A^{-1}K| \le 2\|A^{-1}\| \, |K| \le \lambda^{-1}|K|$$

Hence

$$|h| \le \lambda^{-1}|k| \quad \text{if } k \to 0 \Rightarrow h \to 0$$

Also:

$$\|f'(x) - A\| < \lambda, \quad \forall x \in U$$

$$\Rightarrow \|f'(x) - A\| \, \|A^{-1}\| < \lambda \cdot \frac{1}{2\lambda} < 1$$

$\Rightarrow$ $f'(x)$ is invertible. Let $T = [f'(x)]^{-1}$.

$$\frac{|g(y+k) - g(y) - Tk|}{|k|} = \frac{|x+h-x-Tk|}{|k|}$$

$$= \frac{|h - Tk|}{|k|}$$

$$= \frac{|-T(f(x+h) - f(x) - f'(x)h)|}{|k|}$$

$$\le \frac{\|T\|}{\lambda} \cdot \frac{|f(x+h) - f(x) - f'(x)h|}{|h|}$$

$$\lim_{|k| \to 0} \frac{|g(y+k) - g(y) - Tk|}{|k|} = 0 \Rightarrow g'(y) = T$$

$$\Rightarrow g'(y) = \{f'(g(y))\}^{-1}$$

$\Rightarrow$ g is differentiable at any $y \in V$.

We only need to show that

$$y \mapsto g'(y) \text{ is continuous, } y \in V.$$

We have:

$$g'(y) = \{f'(g(y))\}^{-1}$$

$y \mapsto g(y)$ differ. $\Rightarrow$ $y \mapsto g(y)$ continuous

$f \in \mathscr{C}' \Rightarrow x \mapsto f'(x)$ is continuous, $x \in U$ $\Rightarrow$ composition $f' \circ g$ is continuous.

Inversion is a continuous mapping of Ω on to Ω by Theorem 9.8.

$\Rightarrow$ $y \mapsto g'(y)$, $y \in V$ is continuous

$\Rightarrow$ $g \in \mathscr{C}'$.

Remark: Full assumption $f \in \mathscr{C}'(E)$ only used in last paragraph.

Corollary of (a): let $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$ be $\mathscr{C}'$. $f'(x)$ invertible for every $x \in E$ $\Rightarrow$ f is an open mapping of E into $\mathbb{R}^n$

f need not be 1-1 in E