Def: We say that A and B can be put in 1-1 correspondence, or that A and B have the same cardinal number, or that A and B are equivalent ($A \sim B$) if there exists $f: A \to B$ 1-1 and onto.

This relation satisfies:

(a) It is reflexive:

$A \sim A$

$f: A \to A$

$f(x) = x$ is 1-1 and onto.

(b) It is symmetric:

If $A \sim B$ then $B \sim A$.

If $f: A \to B$ is 1-1 and onto, there exists the inverse function of f, denoted as $f^{-1}: B \to A$ which is also 1-1 and onto.

(c) It is transitive:

If $A \sim B$ and $B \sim C$ then $A \sim C$.

Indeed, if $f: A \to B$ is 1-1 and onto, and $g: B \to C$ is 1-1 and onto, then the composition:

$$g \circ f: A \to C \text{ is 1-1 and onto.}$$

Any relation satisfying (a), (b) and (c) is an equivalent relation.

<u>Definition</u>: For any positive integer n, we define:

$$J_n := \{1, 2, \dots, n\}$$

$$J := \{1, 2, 3, \dots\} = \{\text{set of all positive numbers}\}$$

(a) A is <u>finite</u> if

$$A \sim J_n, \text{ for some } n$$

(b) A is <u>infinite</u> if A is not finite

(c) A is <u>countable</u> if $A \sim J$

(d) A is <u>uncountable</u> if A is neither finite nor countable

(e) A is <u>at most countable</u> if A is finite or countable.

<u>Ex</u>: Let $A = \{0, 1, -1, 2, -2, 3, -3, \dots\}$. Show that A is countable.

<u>Proof</u>: The function

$f: A \to J$ given by:

$$\begin{array}{cccccccccc} & 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \\ f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \end{array}$$

Since $A \sim J \Rightarrow J \sim A$ we can also answer by giving $f: J \to A$, $f(n) = \begin{cases} n/2, & n \text{ even} \\ -\frac{n-1}{2}, & n \text{ odd}. \end{cases}$

Note: The previous example shows that an infinite set can be equivalent to one of its proper subsets. This is not possible for a finite set

Definition: A sequence is a function defined on J:

$$f(n) = x_n, \quad n = 1, 2, \ldots$$

The elements of $\{x_n\}$ can repeat. Any countable set can be arranged in a sequence.

Theorem 2.8: Every infinite subset of a countable set A is countable (i.e.; "countable sets represent the smallest infinity").

Proof: Suppose $E \subset A$, and E is infinite. Arrange the elements of A in a sequence $\{x_n\}$ of distinct elements:

$$x_1, x_2, x_3, x_4, x_5, \ldots$$

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Let n_2 be the smallest integer greater that n_1 such that $x_{n_2} \in E$. We proceed in this way: having chosen $n_1, \ldots, n_{k-1}$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

The function:

$$f: J \to E \text{ given by}$$
$$f(k) = x_{n_k}, \quad k = 1, 2, \dots$$

is 1-1 and onto. Hence E is countable. ∎

The next Theorem shows that not all infinite sets are countable.

Theorem 2.14: Let A be the set of all sequences whose elements are 0 or 1. Then A is uncountable.

Proof:

The elements of A are of this form:

$$1, 0, 1, 0, 1, 1, 1, \dots \in A$$
$$0, 0, 0, 1, 1, 1, 1, \dots \in A.$$

Let $E = \{s_1, s_2, \dots\}$ be a countable subset of A. We can arrange E as follows:

$$\begin{array}{lllll}
s_1: & s_{11} & s_{12} & s_{13} & s_{14} \dots \\
s_2: & s_{21} & s_{22} & s_{23} & s_{24} \dots \\
s_3: & s_{31} & s_{32} & s_{33} & s_{34} \dots \\
 & \vdots & & &
\end{array}$$

From this array, we can construct a sequence that belongs to A as follows:

$$\alpha = \alpha_1, \alpha_2, \alpha_3 \dots$$

If $s_{nn} = 1$ then we define $\alpha_n := 0$
If $s_{nn} = 0$ then we define $\alpha_n := 1$

Clearly, $\alpha \in A$, but $\alpha \notin E$. That is, E is a proper subset of A. We have shown that every countable subset of A is a proper subset of A. Hence, A must be uncountable, for otherwise, A would be a proper subset of itself, which is not possible. ■

Ex: Recall the sets $\mathbb{Q}$ and $\mathbb{I}$ of rational and irrational numbers respectively.
We will show later that $\mathbb{Q}$ is countable.
In order to show that $\mathbb{I}$ is uncountable, from Theorem 2.8, it is enough to show that the interval $[0,1]$ is uncountable
We represent every $x \in [0,1]$ as a decimal:

$x = 0.a_1 a_2 a_3 \dots$, where each a_k denotes one of the digits $0,1,2,3,4,5,6,7,8,9$.

We proceed by contradiction and assume that $[0,1]$ is a countable set. Hence, we can enumerate the elements of $[0,1]$ as:

$$x_1 = 0.a_1 a_2 a_3 \ldots$$
$$x_2 = 0.b_1 b_2 b_3 \ldots$$
$$x_3 = 0.c_1 c_2 c_3 \ldots$$
$$\vdots$$

We form another decimal number $y \in [0,1]$ as follows:

$$y = 0.y_1 y_2 y_3 \ldots$$

where $y_1 \neq 0$, $y_1 \neq 9$ and $y_1 \neq a_1$
$y_2 \neq 0$, $y_2 \neq 9$ and $y_2 \neq b_2$
$y_3 \neq 0$, $y_3 \neq 9$, and $y_3 \neq c_3$, an so on:
$y_n \neq 0$, $y_n \neq 9$, and $y_n \neq c_n$.

Clearly, $0 \leq y \leq 1$. The number y is not one of the numbers with two decimal representations, since $y_n \neq 0, 9$ (Note that $0.1000\ldots$ is the same number as $0.09999\ldots$). Also, $y_n \neq x_n$.

Since we assumed that we had listed all the numbers $0 \leq x \leq 1$ and y is not in the list, we have obtained a contradiction. We conclude that $[0,1]$ is uncountable. ■

Remark: Theorem 2.8 implies that $\mathbb{R}$ is uncountable, for if $\mathbb{R}$ were countable, then $[0,1] \subset \mathbb{R}^n$ would also be countable, which is not true.

- <u>Remark</u>:

We will show in Theorem 2.12 that the union of countable sets is also countable. This fact implies that the infinite set:

$$I \cap [0,1] = \{x \in [0,1] : x \text{ is irrational}\}$$

is uncountable (for otherwise, since $\mathbb{Q}$ is countable, $[0,1]$ would be countable).

Therefore, since $I \cap [0,1]$ is uncountable, Theorem 2.8 implies that the set of irrational numbers I is uncountable.

We will rigorously show later that $\mathbb{Q}$ is countable. However, it is easy to see how $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$ can be put into 1-1 correspondance with $\mathbb{J}$:

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\[1em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\[1em]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\[1em]
\frac{4}{1} & \frac{4}{2} & \cdots & & & & \\[1em]
\frac{5}{1}, & \cdots & & & & &
\end{array}$$