We recall first some operations with sets:

Let A be any set. We can use $\alpha \in A$ to describe a family of sets $\{E_\alpha\}$.

Ex: If $A = J = \{1, 2, 3, \ldots\}$, we would have $E_1, E_2, E_3, \ldots$

Notation: $\iff$ means "if and only if"

$\exists$ means "there exists"

$\forall$ means "for all".

$\Rightarrow$ means "implies"

s.t. means "such that"

Union of sets:

$$S = \bigcup_{\alpha \in A} E_\alpha$$

$$x \in S \iff \exists \alpha \in A \text{ s.t. } x \in E_\alpha$$

Intersection of sets:

$$P = \bigcap_{\alpha \in A} E_\alpha$$

$$x \in P \iff x \in E_\alpha \ \forall \alpha \in A.$$

Ex: If $A = J = \{1, 2, 3, \ldots\}$ we write:

$$S = \bigcup_{i=1}^{\infty} E_i \ , \quad S = \bigcap_{i=1}^{\infty} E_i$$

If $A = J_n$:

$$S = \bigcup_{i=1}^{n} E_i \ , \quad S = \bigcap_{i=1}^{n} E_i$$

Note: If $A \cap B = \emptyset$ then we say that A and B are disjoints.

Ex: Let $A = \{x \in \mathbb{R} : 0 < x \leq 1\}$. We define:
$$E_x = \{y \in \mathbb{R} : 0 < y < x\}, \quad x \in A.$$

Claim: $\bigcup_{x \in A} E_x = E_1$

Clearly, $E_1 \subset \bigcup_{x \in A} E_x$

We need to show that $\left(\bigcup_{x \in A} E_x\right) \subset E_1$:

Let $y \in \bigcup_{x \in A} E_x$.

$\Rightarrow y \in E_x$, for some $x \in A$

$\Rightarrow 0 < y < x \leq 1$

$\Rightarrow 0 < y < 1$

$\Rightarrow y \in E_1$, which gives the desired inclusion

(Recall, two sets are equal, say $P = S$, if and only if $P \subset S$ and $S \subset P$).

Claim $\bigcap_{x \in A} E_x = \emptyset$

Let $y > 0$. Pick $0 < x < y$. Then $y \notin E_x$ and hence $y \notin \bigcap_{x \in A} E_x$. We conclude that:
$$\bigcap_{x \in A} E_x = \emptyset.$$

Properties of unions and intersections:

(a) $A \cup B = B \cup A$ (also, $A \cap B = B \cap A$)

(b) $(A \cup B) \cup C = A \cup (B \cup C)$

[also, $(A \cap B) \cap C = A \cap (B \cap C)$]

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) $A \subset A \cup B$

(e) $A \cap B \subset A$

(f) $A \cup \phi = A$, $A \cap \phi = \phi$

(g) If $A \subset B \Rightarrow A \cup B = B$, $A \cap B = A$.

Proof of (c):

Let $E = A \cap (B \cup C)$ and $F = (A \cap B) \cup (A \cap C)$

The following is true:

$$
\begin{aligned}
x \in E &\iff x \in A \text{ AND } x \in B \cup C \\
&\iff (x \in A \text{ AND } x \in B) \text{ OR } \\
&\qquad (x \in A \text{ AND } x \in C) \\
&\iff x \in A \cap B \text{ or } x \in A \cap C \\
&\iff x \in (A \cap B) \cup (A \cap C) \\
&\iff x \in F.
\end{aligned}
$$

Theorem 2.12: Let $\{E_n\}$, $n=1,2,\ldots$ be a sequence of countable sets, then:

$$S = \bigcup_{n=1}^{\infty} E_n \text{ is also countable}$$

Proof: We arrange every E_n in a sequence $\{x_{nk}\}$, $k=1,2,\ldots$

$$\begin{array}{lccccc}
E_1: & x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\
E_2: & x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\
E_3: & x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\
E_4: & x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\
 & \vdots & & & &
\end{array}$$

All the elements of S can be arranged in a single sequence:

$$x_{11},\ x_{21},\ x_{12},\ x_{31},\ x_{22},\ x_{13},\ x_{41},\ x_{32},\ x_{23},\ x_{14},\ \ldots$$

Since elements can repeat in this sequence, it follows that there is a set $T \subset J$ such that:

$$S \sim T.$$

We note that S is an infinite set because $E_1 \subset S$ and E_1 is an infinite set. Therefore, Theorem 2.8 yields that S is countable.

<u>Corollary</u>: Suppose A is at most countable, and for every $\alpha \in A$, B_α is at most countable. Then:

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is at most countable.

<u>Proof</u>: T is equivalent to a subset of $\bigcup_{i=1}^{\infty} E_n$

<u>Theorem 2.13</u>: Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \dots, a_n)$, where $a_k \in A$, $k = 1, \dots, n$, and the elements $a_1, \dots, a_n$ need not be distinct. Then B_n is countable.

<u>Proof</u>: B_1 is countable since $B_1 = A$. We now proceed by induction. Suppose B_{n-1} is countable $(n \geq 2)$. We note that:

$$B_n = \} (b, a), \ b \in B_{n-1}, \ a \in A \}.$$

For every fixed b:

$$C_b = \{ (b, a) : a \in A \} \sim A \text{ is countable.}$$

Since:

$$B_n = \bigcup_{b \in B_{n-1}} C_b,$$

it follows from Theorem 2.12 that B_n is countable (because it is the union of countable sets).

Corollary: $\mathbb{Q}$ is countable

Proof: Theorem 2.13 yields that:

$$B_2 = \{(a,b) : a, b \text{ are integers}\}$$

is countable. The integers are $A = \{0, 1, -1, 2, -2, \dots\}$. Every element $\frac{p}{q} \in \mathbb{Q}$ can be identified with the pair (p, q). Not all elements of B_2 belong to $\mathbb{Q}$ (i.e; $(2,0)$, $(0,0)$), but we can identify $\mathbb{Q}$ with a subset of B_2. Then, from Theorem 2.8 it follows that $\mathbb{Q}$ is countable. $\blacksquare$

Open and Closed Sets.

<u>Def 2.18</u>. Let X be a metric space

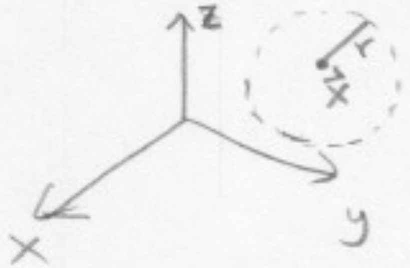
(a) A <u>neighborhood</u> of a point p is a set $N_r(p)$ given by:

$$N_r(p) = \{q \in X : d(p,q) < r\}$$

<u>Ex</u>: If $X = \mathbb{R}^3$ and $\vec{x} \in \mathbb{R}^3$

$$N_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^3 : |\vec{y} - \vec{x}| < r\}$$

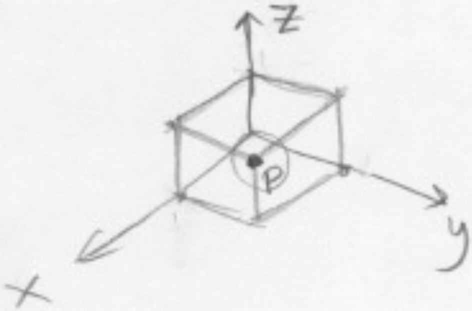
This is a <u>ball</u> of radius r that does not include its boundary.



If $X = \mathbb{R}$, neighborhoods are open intervals.

(b) p is a <u>limit point</u> of the set E if every neighborhood of p contains a point $q \neq p$, $q \in E$

<u>Ex</u>: $X = \mathbb{R}^3$, $E = (0,1) \times (0,1) \times (0,1)$



$p = (1,1,1)$ is a limit point of E

<u>Ex</u>: $X = \mathbb{R}$, $E = \{\frac{1}{n} : n = 1, 2, 3, \dots\}$, $p = 0$ is a limit point of E because every interval $(-r, r)$ contains a point of E, $p = 1$ is not a limit point.

(c) If $p \in E$ and p is not a limit point of E, then p is called an <u>isolated point</u> of E.

<u>Ex</u>: In previous example in (b), $p=1$ is an isolated point.

(d) E is <u>closed</u> if every limit point of E is a point of E.

<u>Ex</u>: $X=\mathbb{R}^3$

$E=\{\vec{x}\in\mathbb{R}^3 : |\vec{x}|\le r\}$ is closed.

<u>Ex</u>: $E=\{\frac{1}{n} : n=1,2,3,\ldots\}$ is not closed, since $0\notin E$

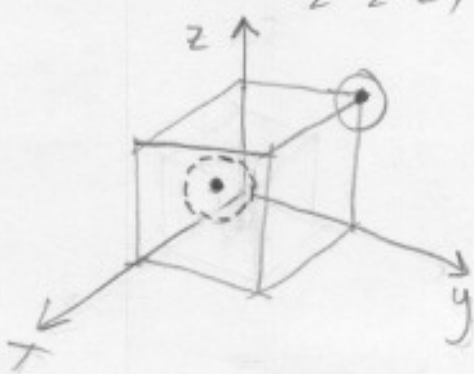
(e) p is an <u>interior point</u> of E if there is a neighborhood $N(p)$ of p such that $N(p)\subset E$

<u>Ex</u>: $X=\mathbb{R}^3$

$E=[0,1]\times[0,1]\times[0,1]$

$p=(1,1,1)$ is not an interior point

$p=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is an interior point



(f) E is <u>open</u> if every point of E is an interior point of E

<u>Ex</u>: $X = \mathbb{R}^3$

Note: if we take $\vec{p} \in B_r(\vec{x})$, no matter how close p is to the boundary, we can find $\tilde{r}$ very small such that $B_{\tilde{r}}(\vec{p}) \subset B_r(\vec{x})$

Every neighborhood $N_r(\vec{x})$ is an <u>open</u> ball, sometimes denoted as $B_r(\vec{x})$.

(g) the <u>complement</u> of E is:

$$E^c = \{p \in X : p \notin E\}$$

(h) E is perfect if E is closed and if every point of E is a limit point of E

<u>Ex</u> : $X = \mathbb{R}^3$

$E = [0,1] \times [0,1] \times [0,1]$ is perfect

$X = \mathbb{R}$, $E = [0,1] \cup \{2\}$ is not perfect since 2 is not a limit point of E.

(i) E is bounded if $\exists M$ and $q \in X$ s.t.

$$d(p,q) < M, \quad \forall p \in E$$

<u>Ex</u> $X = \mathbb{R}^3$

E is bounded.

(j) E is dense in X if every point of X is a limit point of E, or a point of E (or both)

<u>Claim</u>: $\mathbb{Q}$ is dense in $\mathbb{R}$.

<u>Proof</u>: Let $z \in \mathbb{R}$ an irrational number. and let $r > 0$ arbitrary positive number. We need to show that $(z-r, z+r)$ contains a rational number. Let $x := z-r$, $y := z+r$. Since $y-x>0$, the Archimedean property implies that there exists $m \in J$ such that $0 < \frac{1}{m} < y-x$. Again, the Archimedean property yields the existence of $n \in J$ such that

$$n\left(\frac{1}{m}\right) > x, \quad \left(\text{i.e, } \quad x < \frac{n}{m}\right)$$

and we let this n be the least such natural number; that is:

$$\frac{n-1}{m} \leq x < \frac{n}{m}$$

We must have that $\frac{n}{m} < y$, for otherwise we would have $\frac{n-1}{m} \leq x < y \leq \frac{n}{m} \Rightarrow y-x \leq \frac{n}{m} - \frac{n-1}{m} = \frac{1}{m}$

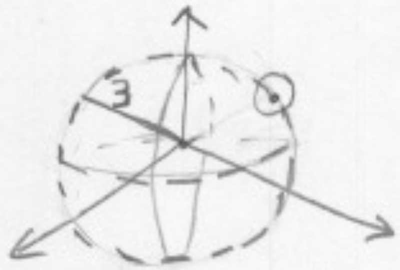
which contradicts that $y-x > \frac{1}{m}$. We conclude;

$$x < \frac{n}{m} < y. \quad \blacksquare$$

<u>Def</u>: X metric space, $E \subset X$.
Define
(a) $E' = \{p \in X : p \text{ is a limit point of } E\}$
(b) The <u>closure</u> of E is defined as:

$$\bar{E} = E \cup E'$$

<u>Ex</u>: $X = \mathbb{R}^3$
$E = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| < 3\}$.



E is a open ball of radius 3.

$$E' = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| \leq 3\}$$

$$\bar{E} = E \cup E' = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| \leq 3\}.$$

In the notation of $\mathbb{R}^k$:

$$\bar{E} = \bar{B}_3(0).$$

<u>Remark</u>: The topological boundary of $E \subset X$ defined as:

$$\partial E = \bar{E} \setminus E^\circ$$

<u>Ex</u>: $E = \{\frac{1}{n} : n = 1, 2, \dots\}$, $\partial E = \{\frac{1}{n} : n = 1, 2, \dots\} \cup \{0\}$

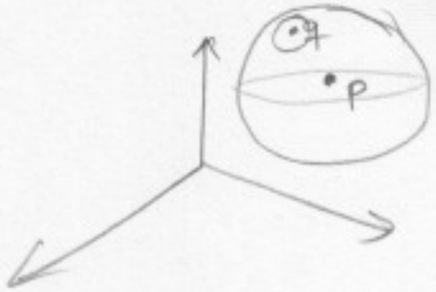
$p \in \partial E$ if for every neighborhood $N_r(p)$:

$N_r(p) \cap E \neq \emptyset$ <u>and</u> $N_r(p) \cap E^c \neq \emptyset$.

Ex. For $E = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| < 3\}$, $\partial E = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| = 3\}$.

<u>Theorem 2.19</u>. Every neighborhood is an open set.

<u>Proof</u>:



Let $E = N_r(p)$ and let $q \in E$. Since $d(p,q) < r$ then $\exists h > 0$ such that:

$$d(p,q) = r - h$$

We consider the neighborhood of q:

$$N_h(q) = \{s : d(s,q) < h\}$$

It remains to show that $N_h(q) \subset E$. Let $s \in N_h(q)$, then:

$$\begin{aligned} d(s,p) &\leq d(s,q) + d(q,p) \\ &< \cancel{h} + r - \cancel{h} = r, \end{aligned}$$

and thus $s \in E$.

<u>Theorem 2.20</u>: If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

<u>Proof</u>: Suppose that there exists $N_r(p)$ that contains only a finite number of points of E, say $q_1, \ldots, q_n$, $q_i \neq p$. Let $r = \min\limits_{1 \leq i \leq n} d(p, q_i)$

Then, $N_r(p) \cap E = \{p\}$ or $\emptyset$, which in any case means that p is not a limit point of E. This a contradiction. ∎

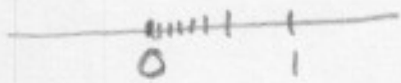
- Examples:

* A finite point set has no limit points.
  Ex: $E = \{2, 4, 6, 8\}$ has no limit points.

* $X = \mathbb{R}^2$ is closed, perfect, open but not bounded.

* $E = \{\text{integers}\} \subset \mathbb{R}$ is closed, not perfect, not bounded

* $E = \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$ is not closed since 0 is a limit point and $0 \notin E$. E is not open since $\frac{1}{2}$ is not an interior point.
(actually, for any n, $\frac{1}{n}$ is not an interior point).



Theorem 2.22: $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$

Proof: Let $A := \left(\bigcup_\alpha E_\alpha\right)^c$, $B = \bigcap_\alpha E_\alpha^c$

Let $x \in A \Rightarrow x \notin E_\alpha \ \forall \alpha \in A \Rightarrow x \in E_\alpha^c \ \forall \alpha \in A$

$\Rightarrow x \in \bigcap_\alpha E_\alpha^c \Rightarrow x \in B \Rightarrow A \subset B$

Let $x \in B \Rightarrow x \in E_\alpha^c \ \forall \alpha \in A \Rightarrow x \notin E_\alpha \ \forall \alpha \in A$

$\Rightarrow x \notin \bigcup_\alpha E_\alpha \Rightarrow x \in \left(\bigcup_\alpha E_\alpha\right)^c \Rightarrow x \in A \Rightarrow B \subset A$

Since $A \subset B$, $B \subset A \Rightarrow A = B$.