Theorem 2.23:

E is open $\underbrace{\text{if and only if}}_{\Longleftrightarrow}$ E^c is closed.

Proof:

We need to prove two directions:

($\Longleftarrow$) suppose E^c is closed. Let $x \in E$. Since $x \notin E^c$ and E^c is closed, by definition of a closed set, it follows that x is not a limit point of E^c. Hence, there exists $N(x)$ such that $N(x) \cap E^c = \emptyset$; that is, $N(x) \subset E$. Thus, x is an interior point of E. Since x was arbitrary, we conclude that E is open.

Ex: $E^c = \{|\vec{x}| \geq 1\}$ closed

$E = \{|\vec{x}| < 1\}$ open

($\Longrightarrow$) Suppose E is open. Let x be a limit point of E^c. Then, for every neighborhood $N(x)$ of x, $N(x) \cap E^c$ contains a point of E^c. Since E is open, x can not be in E, for otherwise x would be an interior point of E, and a neighboorhood $N(x)$ contained in E would have to exist. Hence $x \in E^c$. We have shown that all the limit points of E^c are in E^c. Hence, E^c is closed.

Theorem 2.24:

(a) For any collection $\{G_\alpha\}$ of open sets,

$$\bigcup_\alpha G_\alpha \text{ is open}$$

(b) For any collection $\{F_\alpha\}$ of closed sets

$$\bigcap_\alpha F_\alpha \text{ is closed}$$

(c) For any finite collection $G_1, \dots, G_n$ of open sets, $\bigcap_{i=1}^{n} G_i$ is open.

Note: (c) is not true for an infinite collection. For example,

Let $I_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$, I_i is open, $i = 1, 2, \dots$,

but

$$\bigcap_{i=1}^{\infty} I_i = \{0\}, \text{ which is not open.}$$

(d) For any finite collection $F_1, \dots, F_n$ of closed sets, $\bigcup_{i=1}^{n} F_i$ is closed

Note: (d) is not true for an infinite collection. For example,

Let $I_i = [-i, i]$, I_i is closed, $i = 1, 2, \dots$,

but

$$\bigcup_{i=1}^{\infty} I_i = \mathbb{R} \text{ is open.}$$

Proof:

The proof of (a) and (c) are left to the reader.

For (b), $\left(\bigcap_{\alpha} F_\alpha\right)^c = \bigcup_{\alpha} F_\alpha^c$ is open, and hence $\bigcap_{\alpha} F_\alpha$ is closed.

For (d), $\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c$ is open, and hence $\bigcup_{i=1}^{n} F_i$ is closed.

Theorem 2.27: Let X be a metric space and $E \subset X$. Then:

(a) $\bar{E}$ is closed

(b) $E = \bar{E} \iff E$ is closed

(c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$ (i.e., $\bar{E}$ is the smallest closed set that contains E).

Proof:

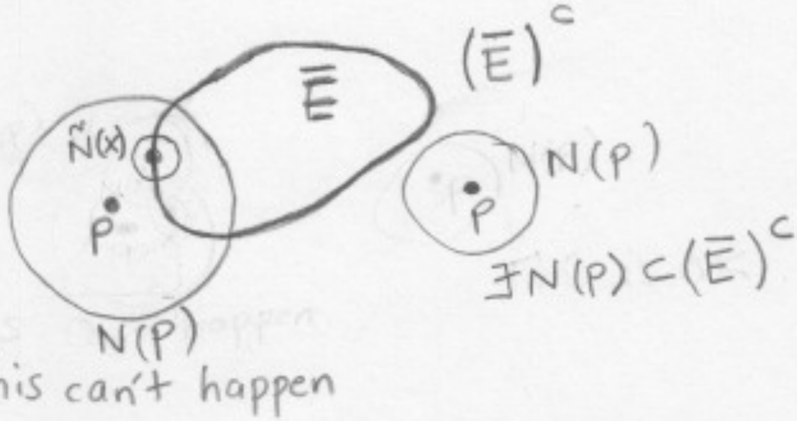
(a) We show that $(\bar{E})^c$ is open. Let $p \in (\bar{E})^c$. Since $\bar{E} = E \cup E'$ and $p \notin \bar{E}$, the p is neither a point of E nor a limit point of E. Thus, there exists $N(p)$ such that $\underline{N(p) \cap E = \emptyset}$. We also have that $N(p) \cap E' = \emptyset$, for

- otherwise, if $x \in E' \cap N(p)$, then we could find $\tilde{N}(x) \subset N(p)$ such that $\tilde{N}(x) \cap E \neq \emptyset$.

We conclude that $\underline{N(p) \cap \bar{E} = \emptyset}$; i.e:

$$N(p) \subset (\bar{E})^c,$$

which shows that $(\bar{E})^c$ is open.



(b) ($\Leftarrow$) If E is closed, then $E' \subset E$ and hence

$$\bar{E} = E \cup E' = E$$

($\Rightarrow$) If $E = \bar{E}$, since $\bar{E}$ is closed, then E is closed.

(c) Note that $E \subset F$ implies that:

$$E' \subset F'$$

Since F is closed then $F' \subset F$.

Hence, $\bar{E} = E \cup E' \subset F$.

<u>Theorem 2.28</u> : Let $E \subset \mathbb{R}$, $E \neq \emptyset$, E bounded above. Let $y \in \text{Sup } E$. Then $y \in \bar{E}$.

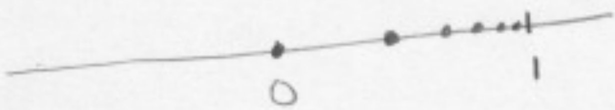
<u>Proof</u>: If $y \in E$, clearly, $y \in \bar{E}$. Assume $y \notin E$. Then for every $h > 0$, there exists <u>$x_h \in E$</u> such that

$$x_h \in (y-h, y),$$

for otherwise $y-h$ would be an upper bound of E. Hence, $y \in E'$, which yields that $y \in \bar{E}$.

<u>Ex</u>: $E = \{1 - \frac{1}{n}, n = 1, 2, \dots\}$

$\text{Sup } E = 1$, $1 \notin E$, $1 \in E'$.



<u>Note</u>: From Theorem 2.28, if $y = \text{Sup} E$ and E is closed, then $y \in E$.