We will sometimes use the following:

Definition: Let X be a metric space and let

$$E \subset Y \subset X.$$

(Y is also a metric space with the same distance as in X).

We say that E is open relative to Y if for every $p \in E$, $\exists r > 0$ such that:

$$S_r(p) := \{q \in Y : d(p,q) < r\} \subset E$$

Note: $S_r(p)$ might not be open in X

Example: Consider $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $E = (a,b)$, then E is open relative to Y.

We have the following:

Theorem 2.30: Suppose $Y \subset X$. Then:

$E \subset Y$ is open relative to $Y \iff E = Y \cap G$, for some open set G of X.

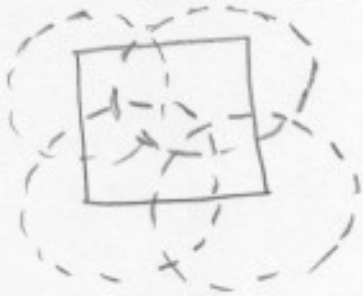
Compact sets

Def: Let X be a metric space, $E \subset X$. An open cover of E is a collection $\{G_\alpha\}$ of open subsets of X such that:

$$E \subset \bigcup_\alpha G_\alpha$$

Ex $X = \mathbb{R}^2$

$E = [0,1] \times [0,1]$



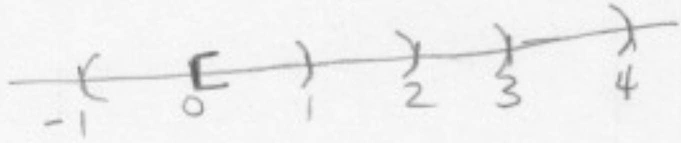
An example of an open cover of E

Def: Let $K \subset X$. K is said to be compact if every open cover of K contains a finite subcover. That is, if $\{G_\alpha\}$ is an open cover of K, there exists $\alpha_1, \dots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$$

Ex: Show that $E = \{x \in \mathbb{R} : x \geq 0$ is not compact.

$E = [0, \infty)$, $X = \mathbb{R}$



Let $G_n = (-1, n)$. Then $E = \bigcup_{n=1}^{\infty} G_n$, but this open cover has no finite subcover.

Ex: Show that the closed interval $I=[0,1]$ is compact.

Proof: Let $G=\{G_\alpha\}$ be an open cover of I. Since $0\in I$ then there exists α such that $0\in G_\alpha$. Since G_α is open, $\exists\, \delta>0$ s.t. $[0,\delta]\subset G_\alpha$. Hence $\delta\in E$ where:

$$E:=\{x\in I: [0,x] \text{ is contained in the union of a finite number of sets in } G\}$$

Since $E\neq\emptyset$, and E is bounded above, $\sup E$ exists. Let:

$$x^*=\sup E$$

We will show that $x^*=1$. Since 1 is an upper bound of E, and x^* is the smallest upper bound of E; it follows that $x^*\leq 1$. We assume first that $x^*<1$, and proceed as follows:

Since $x^*\in[0,1]\subset\bigcup_\alpha G_\alpha$, then x^* belong to one of the sets in the collection, say G_0.

Since G_0 is open and x^* is an interior point of G_0, there exists $\tilde{\varepsilon}>0$ such that:

$$(x^*-\tilde{\varepsilon},\ x^*+\tilde{\varepsilon})\subset G_0$$

$$\Rightarrow [x^*-\varepsilon, x^*+\varepsilon]\subset(0,1)\subset G_0, \text{ for } \varepsilon \text{ small enough.}$$

Since $x^*=\sup E$, there exists $\tilde{x}\in E$ such that $x^*-\varepsilon\leq\tilde{x}<x^*$, for otherwise $x^*-\varepsilon$ would be an upper bound of E, and x^* is the smallest upper bound of E. Hence, $[0,\tilde{x}]$ can be covered with a finite number of sets in G. Since $[0,x^*-\varepsilon]\subset[0,\tilde{x}]$, it follows that $[0,x^*-\varepsilon]$ can also be covered with a finite number of sets in G;

Hence:

$$[0, x^* - \varepsilon] \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$$

By adding G_0 to this finite collection we obtain:

$$[0, x^* + \varepsilon] \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \cup G_0,$$

Since $[x^* - \varepsilon, x^* + \varepsilon] \subset G_0$.

This implies that $x^* + \varepsilon \in E$, which contradicts that x^* is an upper bound of E. From this contradiction we conclude that $x^* = 1$. (since $x^* \leq 1$, and $x^* < 1$ is not true).

Now, since $1 \in [0,1] \subset \bigcup_\alpha G_\alpha$, then 1 belongs to one of these open sets, say G_2. Since 1 is an interior point of G_2, there exists $0 < \varepsilon < \frac{1}{4}$ such that $(1-\varepsilon, 1+\varepsilon) \subset G_2$. Since $1 = \sup E$, there exists $\tilde{x} \in E$ s.t:

$$1 - \varepsilon \leq \tilde{x} < 1,$$

for otherwise $1-\varepsilon$ would be a smaller upper bound of E, which contradicts that 1 is the smallest upper bound of E.

Since $\tilde{x} \in E$, $[0, \tilde{x}]$ can be covered with a finite number of open sets in G, say:

$$[0, \tilde{x}] \subset G_{\beta_1} \cup G_{\beta_2} \cup \cdots \cup G_{\beta_m}$$

Adding G_2 to this collection we conclude:

$$[0, 1] \subset G_{\beta_1} \cup \cdots \cup G_{\beta_m} \cup G_2 \text{ ; since } (1-\varepsilon, 1] \subset G_2 \text{ and } \tilde{x} \geq 1 - \varepsilon.$$

Therefore $[0,1]$ is compact.

<u>Theorem 2.34</u>: Let $K \subset X$, K compact. Then K is closed.

<u>Proof</u>: We will show that K^c is open. Let $p \in K^c$. We need to show that p is an interior point of K^c; that is, that $p \in (K^c)^\circ$. For every $q \in K$, we define:

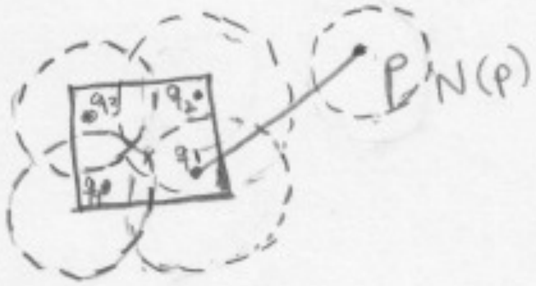
$$r_q := \frac{1}{4} d(p, q)$$

Clearly:

$$K \subset \bigcup_{q \in K} N_{r_q}(q)$$

Since K is compact, it can be covered with a finite number of neighborhoods:

$$K \subset \underbrace{N_{r_{q_1}}(q_1) \cup N_{r_{q_2}}(q_2) \cup \dots \cup N_{r_{q_n}}(q_n)}_{\text{Denote this open set as } W.}$$



$$\text{Let } V := N_{r_{q_1}}(p) \cap \dots \cap N_{r_{q_n}}(p)$$

The open set V contains a neighborhood of p, $N(p)$, such that:

$$N(p) \cap W = \emptyset$$

Indeed, if $s \in N(p)$, then:

$$d(p, q_i) \leq d(q_i, s) + d(s, p) \Rightarrow d(s, q_i) \geq d(p, q_i) - d(s, p) \geq 4r_{q_i} - r_{q_i} > r_{q_i}$$

$$\Rightarrow s \notin N_{r_{q_i}}(q_i),\ i = 1, \dots, n \Rightarrow N(p) \cap K = \emptyset \Rightarrow p \in (K^c)^\circ. \ \blacksquare$$

Theorem 2.35: Let $K \subset X$, K compact.
If $F \subset K$ and F is closed, then F is compact.

Proof: Let $\{G_\alpha\}$ be an open cover of F

Since:

$$F \subset K \subset X \quad \text{and} \quad F^c \text{ is open}$$

then

$$\{G_\alpha\} \cup F^c \text{ is a cover of } K.$$

Since K is compact, we have a finite subcover of K, say:

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n} \cup F^c$$

Hence; F can be also cover with this finite collection of open sets.

$$F \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$$

$\Rightarrow$ F is compact

Corollary: If F is closed and K compact. Then $F \cap K$ is compact.

Proof: K compact $\Rightarrow$ K is closed
$\Rightarrow$ $F \cap K$ is closed; since intersection of closed sets is closed

Since $F \cap K \subset K$, previous theorem $\Rightarrow$ $F \cap K$ is compac

Notation: The open interval (a, b) is also denoted as segment.
The closed interval $[a,b]$ is just denoted as interval.
$[a, b)$ is called a half open interval.

Theorem 2.36: If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non-empty, then

$$\bigcap_\alpha K_\alpha \text{ is not empty.}$$

Proof: Fix a member K_1 of $\{K_\alpha\}$. Let $G_\alpha = K_\alpha^c$. We proceed by contradiction and assume that

$$\bigcap_\alpha K_\alpha = \phi.$$

Hence:

$$K_1 \cap \Big(\bigcap_{\substack{\alpha \\ \alpha\neq 1}} K_\alpha \Big) = \phi$$

Hence $K_1 \subset \Big(\bigcap_{\substack{\alpha \\ \alpha\neq 1}} K_\alpha \Big)^c$,

That is:

$$K_1 \subset \Big(\bigcup_{\substack{\alpha \\ \alpha\neq 1}} G_\alpha \Big);$$ recall formula $(\bigcap A_\alpha)^c = \bigcup A_\alpha^c$

Since K_1 is compact, $\exists \alpha_1, \dots, \alpha_n$ s.t:

$$K_1 \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}, \quad \alpha_i \neq 1$$

This says that K_1 is contained in the union of all the complements of $K_{\alpha_1}, \dots, K_{\alpha_n}$, which means:

$$K_1 \cap K_{\alpha_2} \cap \dots \cap K_{\alpha_n} = \emptyset,$$

which contradicts that any finite intersection of the collection is non-empty. $\boxtimes$

<u>Corollary</u>: If we have:

$$K_1 \supset K_2 \supset K_3 \supset \cdots,$$

with $K_i \neq \emptyset$, K_i compact, $i = 1, 2, \dots$, then

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset \quad \boxtimes$$

<u>Theorem 2.37</u>: If E is an infinite subset of a compact set K, then E has a limit point in K.

<u>Proof</u>:

If this is not true, then for each $q \in K$, $\exists N(q)$ s.t.

$$N(q) \cap E = \{q\} \quad \text{if } q \in E$$
$$\text{or } N(q) \cap E = \emptyset \quad \text{if } q \notin E$$

Hence, we have obtained an open cover of K:

$$K \subset \bigcup_{q} N(q)$$

Since K is compact, $\exists \{q_1, \dots, q_n\}$ such that:

$$K \subset N(q_1) \cup \cdots \cup N(q_n)$$

Since $E \subset K$, then

$$E \subset N(q_1) \cup \cdots \cup N(q_n),$$

but this is not possible since:

$E \cap N(q_i) = \emptyset$ or q_i, $i = 1, \dots, n$, and E is an infinite set. ∎