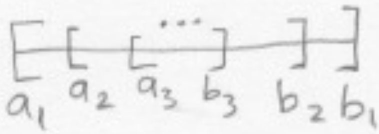
Theorem 2.38 : If we have:

$$I_1 \supset I_2 \supset I_3 \supset \dots$$

where each I_n an interval in $\mathbb{R}$ (i.e., closed interval) then:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Proof : Let $I_n = [a_n, b_n]$



and define:

$$x = \sup \{a_n : n = 1, \dots\}.$$

Let $E = \{a_n : n = 1, \dots\}$

We will show that $x \in I_m$; $m = 1, 2, \dots$

We have, for any n, m:

$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m \qquad (1)$$

If we fix m and let $n = 1, 2, \dots$, then (1) implies that b_m is an upper bound of E. Since $x = \sup E$ is the smallest upper bound of E we obtain:

$$x \leq b_m , \qquad (2)$$

The previous argument holds for every $m = 1, 2, 3, \dots$ and hence (2) holds for every $m = 1, 2, 3, \dots$

Hence, since $x = \sup E$:

$$a_m \leq x \leq b_m , \text{ for all } m = 1, 2, \dots ,$$

that is, $x \in [a_m, b_m] = I_m$, for all $m = 1, 2, \dots$

$$\Rightarrow x \in \bigcap_{m=1}^{\infty} I_m .$$

Theorem 2.39. If

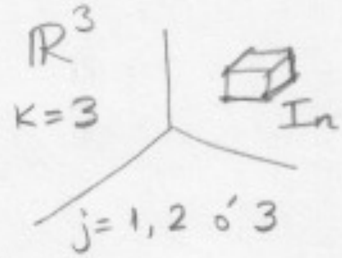
$I_1 \supset I_2 \supset I_3 \ldots,$

where $I_n \in \mathbb{R}^K$, $I_n = [a_{n,1}, b_{n,1}] \times [a_{n,2}, b_{n,2}] \times \cdots [a_{n,K}, b_{n,K}]$

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Proof: For every $j = 1, \ldots, K$,

$$[a_{1,j}, b_{1,j}] \supset [a_{2,j}, b_{2,j}] \supset [a_{3,j}, b_{3,j}] \supset \cdots$$



Theorem 2.38 implies that:

$\exists\, x_j^*$, $1 \le j \le K$ such that:

$$a_{n,j} \le x_j^* \le b_{n,j} \quad , \quad 1 \le j \le K$$

If we define $x^* := (x_1^*, \ldots, x_K^*)$ then;

$$x^* \in I_n, \; n = 1, 2, \ldots \;\blacksquare$$

<u>Def</u>: A k-cell $I \subset \mathbb{R}^k$ is defined as:

$$I = \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{R}^k;\quad a_i \le x_i \le b_i,\ 1 \le i \le k\}$$
$$= [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k]$$

<u>Ex</u>. $[a, b]$ 1 - cell

 2 - cell

 3 - cell

<u>Theorem 2.40</u> Every k-cell I is compact.

<u>Proof</u>: Let $I = [a_1, b_1] \times \dots \times [a_k, b_k]$ and define:

$$\delta := \sqrt{\sum_{i=1}^{k} (b_i - a_i)^2}$$

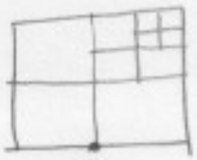
(Note: δ is the length of the diagonal of the square or cube in $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively).

We have:

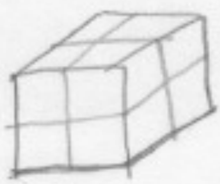
$$|\vec{x} - \vec{y}| < \delta, \quad \forall x, y \in I.$$

We proceed by contradiction. Suppose that $\exists \{G_\alpha\}$ an open cover of I which contains no finite subcover of I. Let

$$c_i := \frac{a_i + b_i}{2}$$



c_i



<u>Ex</u>: $2^k = 2^3 = 8$ k-cells

We can split I into 2^k k-cells:

$$I = \bigcup_{i=1}^{2^k} I_i$$

This partition is given by the intervals:

$$[a_i, c_i] \text{ and } [c_i, b_i], \quad i = 1, \dots, k$$

At least one of these I_i, say I_1, can not be covered by any finite subcollection of $\{G_\alpha\}$ (for otherwise I could be covered by a finite subcollection). We subdivide I_1 in the same way to obtain a sequence:

(a) $I \supset I_1 \supset I_2 \supset I_3 \supset \dots$

(b) I_n is not covered by any finite subcollection of $\{G_\alpha\}$.

(c) $x, y \in I_n \Rightarrow |x-y| \leq \frac{\delta}{2^n}$.

Theorem 2.3 $\Rightarrow \exists\, x^* \in \bigcap_{n=1}^{\infty} I_n$. Since $x^* \in I \subset \bigcup_\alpha G_\alpha$, then $\exists \alpha$ s.t. $x^* \in G_\alpha$. Since G_α is open, $\exists\, B_r(x^*) \subset G_\alpha$ for some r.

For N large enough:

$$\frac{\delta}{2^N} < r,$$

which implies $I_N \subset B_r(x^*) \subset G_\alpha$, contradicting (b). We conclude that I can be covered with a finite collection in $\{G_\alpha\}$. ∎

Bounded Sets

Recall the definition of a bounded set in a metric space X:

Def: $E \subset X$, X metric space.

E is bounded if $\exists M > 0$ and $q \in X$ s.t.:

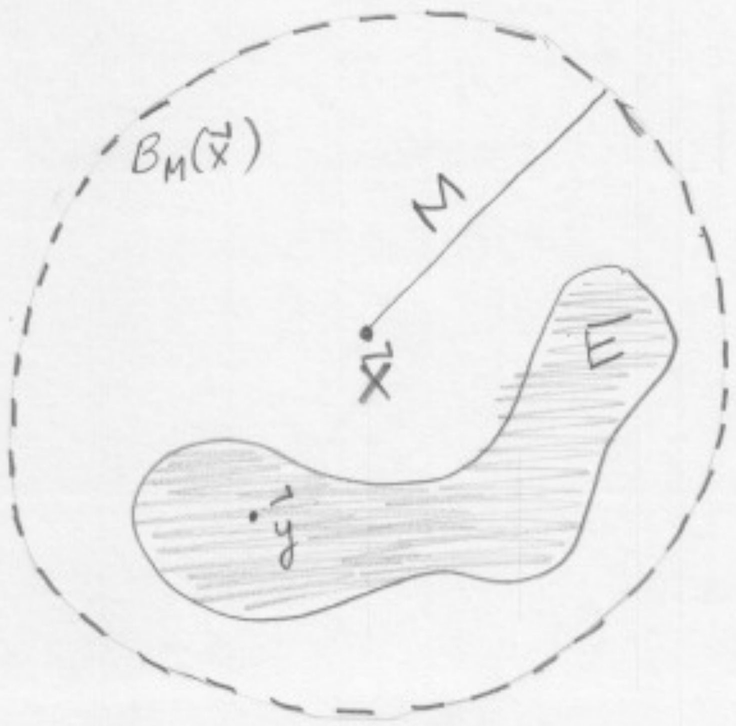
$$d(p,q) < M, \quad \forall p \in E$$

If $X = \mathbb{R}^k$, the $E \subset \mathbb{R}^k$ is bounded if $\exists M > 0$ and $\vec{x} \in \mathbb{R}^k$ s.t.

$$|\vec{y} - \vec{x}| < M, \quad \forall \vec{y} \in E.$$

This means that E is contained in the open ball of radius M, centered at $\vec{x}$:

$$B_M(\vec{x}) = \{\vec{z} \in \mathbb{R}^k : |\vec{z} - \vec{x}| < M\}.$$



<u>Theorem 2.41</u>: Let $E \subset \mathbb{R}^k$. Then the following are equivalent:

(a) E is closed and bounded.

(b) E is compact

(c) Every infinite subset of E has a limit point in E.

<u>Note</u>: The equivalence of (a) and (b) is called the Heine-Borel Theorem.

<u>Proof</u>: <u>(a) $\Rightarrow$ (b).</u>

If E is bounded, then E is contained in some k-cell I:

$I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ is compact

Since $E \subset I$, E is closed and I compact, then Theorem 2.35 implies that E is compact.

<u>(b) $\Rightarrow$ (c)</u>

This implication is proven in Theorem 2.37

<u>(c) $\Rightarrow$ (a)</u>

We assume that every infinite subset of E has a limit point in E.

We proceed by contradiction.

If E is not bounded, then E is not contained in $B_1(\vec{0})$, hence $\exists\, \vec{x}_1 \in E$ such that

$$|\vec{x}_1| > 1, \text{ i.e.; } \vec{x}_1 \notin B_1(\vec{0})$$

Since E is not contained in $B_2(\vec{0})$ then $\exists\, \vec{x}_2 \in E$ such that:

$$|\vec{x}_2| > 2 \text{ ; i.e.; } \vec{x}_2 \notin B_2(\vec{0}).$$

We can choose $\vec{x}_2 \neq \vec{x}_1$; for otherwise E would be bounded. Proceeding is this way, since E is not contained in $B_n(\vec{0})$ then $\exists\, \vec{x}_n \in E$, $\vec{x}_n \neq \vec{x}_i$, $i = 1, \ldots, n-1$, such that:

$$|\vec{x}_n| > n \text{ ; i.e.; } \vec{x}_n \notin B_n(\vec{0}).$$

The set $S = \{\vec{x}_n : n = 1, 2, \ldots\}$ is infinite but clearly has no limit point in $\mathbb{R}^k$ (and hence no limit point in E). Indeed if $\vec{y}$ were a limit point of S, then given any N, $B_N(\vec{y})$ would have to contain infinitely many elements of S, but this is not possible since $B_N(\vec{y}) \subset B_{N+|\vec{y}|}(\vec{0})$ and for all $n > N + |\vec{y}|$, $\vec{x}_n \notin B_{N+|\vec{y}|}(\vec{0})$.

Since we have contradicted assumption (c) we conclude that E is bounded.

- If E is not closed then $\exists \vec{x}_0 \in E'$ such that $\vec{x}_0 \notin E$. By definition of limit point, $\exists \ \vec{x}_1 \in B_1(\vec{x}_0)$, $\vec{x}_1 \in E$, $\vec{x}_1 \neq \vec{x}_0$:

$$\Rightarrow \quad |\vec{x}_1 - \vec{x}_0| < 1$$

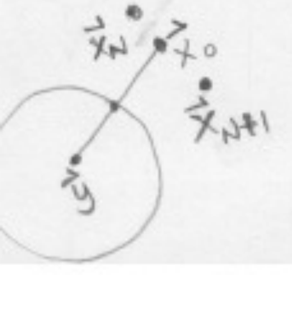
Again, since $\vec{x}_0$ is a limit point of E, $\exists \vec{x}_2 \in B_{\frac{1}{2}}(\vec{x}_0)$ such that $\vec{x}_2 \in E$, $\vec{x}_2 \neq \vec{x}_0$, $\vec{x}_2 \neq \vec{x}_1$:

$$|\vec{x}_2 - \vec{x}_0| < \frac{1}{2}$$

Proceeding in this way, for any n, since $\vec{x}_0$ is a limit point of E, $\exists \vec{x}_n \in B_{\frac{1}{n}}(\vec{x}_0)$ such that $\vec{x}_n \in E$, and we can choose $\vec{x}_n$ so that $\vec{x}_n \neq \vec{x}_i$, $i = 1, .., n-1$, for otherwise $\vec{x}_0$ would not be a limit point (recall that any neighborhood of $\vec{x}_0$ must contain infinitely many elements of E):

$$\Rightarrow \quad |\vec{x}_n - \vec{x}_0| < \frac{1}{n}, \quad n = 1, 2, \dots$$

The set $S = \{\vec{x}_n : n = 1, 2, ..\}$ is an infinite set. We show next that $\vec{x}_0$ is the only limit point of S in $\mathbb{R}^k$. For if $\vec{y} \in \mathbb{R}^k$, $\vec{y} \neq \vec{x}_0$ we can estimate:



$$|\vec{y} - \vec{x}_0| \leq |\vec{y} - \vec{x}_n| + |\vec{x}_n - \vec{x}_0|$$

$$\Rightarrow |\vec{y} - \vec{x}_n| \geq |\vec{y} - \vec{x}_0| - |\vec{x}_n - \vec{x}_0| \geq |\vec{x}_0 - \vec{y}_0| - \frac{1}{n}$$

$$\Rightarrow |\vec{y} - \vec{x}_n| \geq \frac{1}{2} |\vec{x}_0 - \vec{y}_0|, \text{ for } n \geq N, \ N \text{ large enough.}$$

The previous inequality says that $\vec{y}$ can not be a limit point of S.

$$\Rightarrow \quad S' \cap E = \phi,$$

which contradicts S must have a limit point in E (hypothesis (c)).

From this contradiction we conclude that E is closed. ∎

<u>Remark</u>: Note the Previous theorem is proved for $X = \mathbb{R}^k$. For any metric space X, (b) $\Leftrightarrow$ (c) is still true, but in general, (a) $\not\Rightarrow$ (b) and (a) $\not\Rightarrow$ (c).

<u>Theorem 2.42</u>: (Weierstrass).
Every bounded infinite subset E of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

<u>Proof</u>:
Since E is bounded $\Rightarrow \exists I$, a k-cell such that $E \subset I$.

Since I is compact, previous theorem (actually Theorem 2.37) implies that E has a limit point in I.