Perfect sets

Recall that $E \subset X$ is a perfect set if E is closed and if every point of E is a limit point of E.

Ex : Let $X = \mathbb{R}^2$. The following sets in $\mathbb{R}^2$ are perfect:

1) $\mathbb{R}^2$ is perfect
2) Any 2-cell is perfect: 
3) Any $E \subset \mathbb{R}^2$ finite set is not perfect. For example, $E = \{(0,1), (1,2), (2,3)\} \subset \mathbb{R}^2$ is not perfect, since every point in E is not a limit point. All point in E are isolated points.
4) $E = \{(m, 0) : m \text{ is an integer}\}$ is not perfect.
5) $E = I \cup \{(3,5)\}$, $I = [0,1] \times [0,1]$ is not perfect

We have:

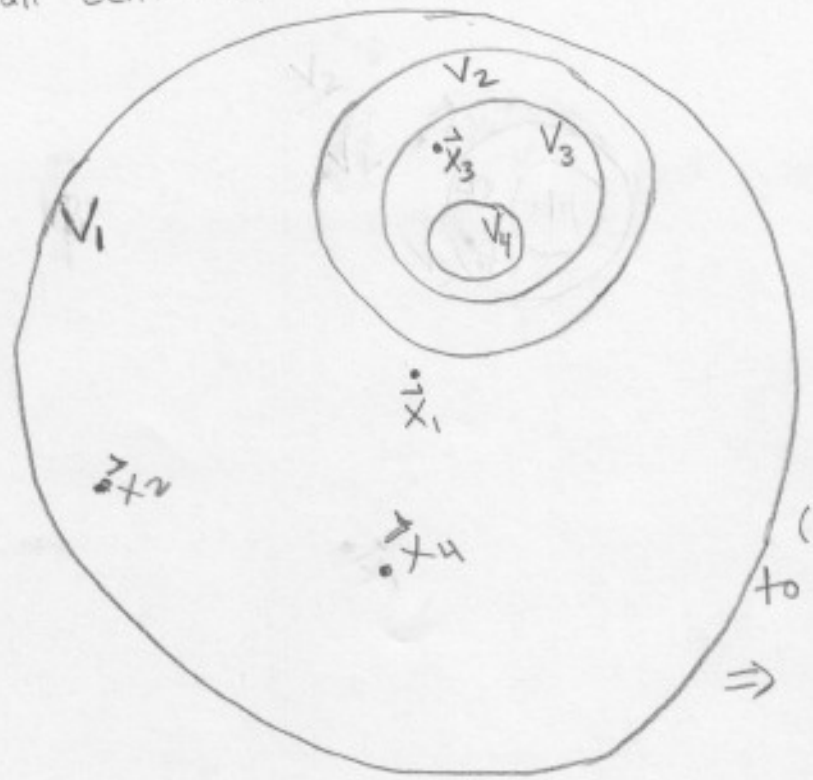
Theorem 2.4 : Let $P \neq \emptyset$ be a perfect set in $\mathbb{R}^k$. Then P is uncountable.

Proof: Since P has limit points, P is not a finite set, hence P is an infinite set. We proceed by contradiction and assume that P is countable; i.e.

$$P = \{\vec{x}_1, \vec{x}_2, \dots\}$$

Every $\vec{x}_i$, $i = 1, 2, \dots$ is a limit point.

The idea of the proof is the following: since every point x_i, $i=1,2,\dots$ is a limit point, we can find a sequence of open balls, nested and denoted as $V_1, V_2, \dots$, in such a way that $\vec{x}_n$ does not belong to the open ball V_{n+1}. We start with V_1 an open ball centered at $\vec{x}_1$.



Notes:
$\vec{x}_1 \notin \overline{V}_2$
$\vec{x}_2 \notin \overline{V}_3$
$\vdots$
$\vec{x}_n \notin \overline{V}_{n+1}$

The points $\vec{x}_2, \vec{x}_3, \dots$ do not have to belong (although they could belong) to the balls $\overline{V}_2, \overline{V}_3, \dots$

$\Rightarrow \overline{V}_1 \supset \overline{V}_2 \supset \overline{V}_3 \supset \dots$

$\Rightarrow$ Let $K_n = \overline{V}_n \cap P$

K_n, $n=1,2,\dots$ is compact since it is a closed set inside the compact set $\overline{V}_n$. We have:

$$K_1 \supset K_2 \supset K_3 \supset \dots$$

Since $\vec{x}_n \notin V_{n+1} \Rightarrow \bigcap_{n=1}^{\infty} K_n = \emptyset$,

but this contradicts the fact that, when we have a nested sequence of non-empty compact sets as above:

$\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. From this contradiction we conclude that P is uncountable

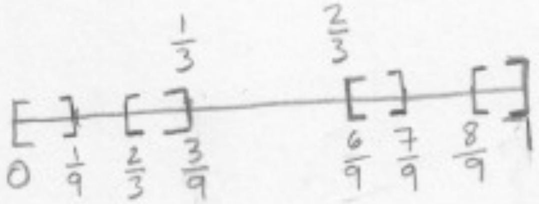
The Cantor Set:

Let $E_0 = [0,1]$. Remove $(\frac{1}{3}, \frac{2}{3})$ and let:

$$E_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$

Remove the middle thirds of these intervals, and let:

$$E_2 := [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{3}{9}] \cup [\tfrac{6}{9}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$



We continue in this way and we obtain a sequence of compact sets E_n such that:

(a) $E_1 \supset E_2 \supset E_3 \supset \cdots$

(b) E_n is the union of 2^n intervals, each of length $\frac{1}{3^n}$.

<u>Define</u>: $$P := \bigcap_{n=1}^{\infty} E_n$$

From Theorem 2.36 it follows that <u>$P \neq \emptyset$</u>.

From Theorem 2.24 we have that <u>P is closed</u>.

Clearly, P is bounded.

Hence, Heine-Borel theorem gives that:

<u>P is compact</u>.

Remark: For any $m \in \{1, 2, \ldots\}$ and $k \in \{0, 1, 2, \ldots\}$

$$\Rightarrow \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subset P^c$$

Indeed; if $m=2$, and $k=1$:

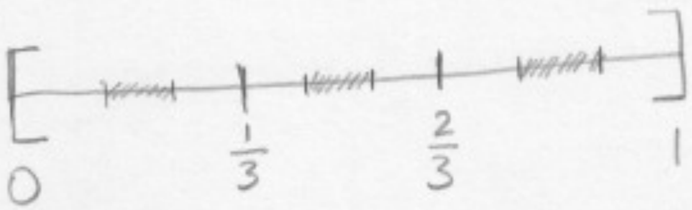
$$\left(\frac{4}{9}, \frac{5}{9}\right) \subset P^c$$

For $m=2$, $k=2$:

$$\left(\frac{7}{9}, \frac{8}{9}\right) \subset P^c$$

If $m=2$, $k=0$:

$$\left(\frac{1}{9}, \frac{2}{9}\right) \subset P^c$$

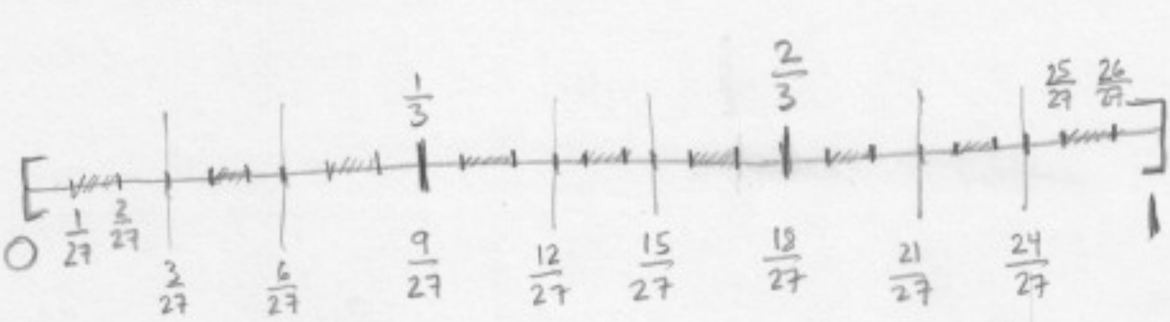


If $m=3$, for $k = 0, 1, 2, 3, 4, 5, 6, 7, 8$

$\left(\frac{1}{27}, \frac{2}{27}\right) \subset P^c$, $\left(\frac{4}{27}, \frac{5}{27}\right) \subset P^c$, $\left(\frac{7}{27}, \frac{8}{27}\right) \subset P^c$,

$\left(\frac{10}{27}, \frac{11}{27}\right) \subset P^c$, $\left(\frac{13}{27}, \frac{14}{27}\right) \subset P^c$, $\left(\frac{16}{27}, \frac{17}{27}\right) \subset P^c$,

$\left(\frac{19}{27}, \frac{20}{27}\right) \subset P^c$, $\left(\frac{22}{27}, \frac{23}{27}\right) \subset P^c$, $\left(\frac{25}{27}, \frac{26}{27}\right) \subset P^c$



Theorem: P does not contain any open interval.

Proof: We proceed by contradiction and assume that $(\alpha, \beta) \subset P$. For m large enough:

$$\beta - \alpha > \frac{6}{3^m},$$

and therefore (α, β) must contain a segment of the form:

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right), \qquad (1)$$

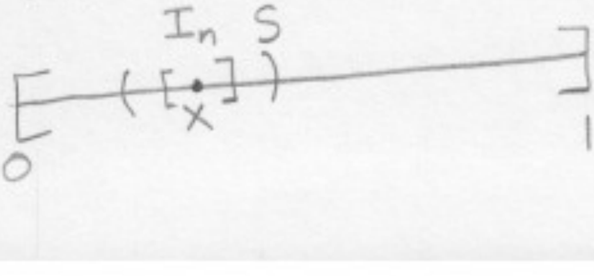
which is a contradiction, since we have seen that segments of the form (1) are contained in P^c

Theorem: P is perfect.

Proof: We have shown that P is closed. We need to show that every point of P is a limit point of P. Let $x \in P$ and let S be any open interval such that $x \in S$.

Let I_n be the interval of E_n which contains x (recall that $x \in \bigcap_{n=1}^{\infty} E_n$, $E_n = \bigcup_{i=1}^{2^n} I_i$).

For n large enough, $I_n \subset S$. This implies that $x \in P'$. ∎



Remark: Since P is perfect, P is uncountable.