# Chapter 4
# Measure theory

<u>Def</u>: A function $\varphi$ defined for every subset $A$ of an arbitrary set $X$ is called an <u>outer measure</u> on $X$ if:

(i) $\varphi(\emptyset) = 0$

(ii) $0 \le \varphi(A) \le \infty, \quad \forall A \subset X$

(iii) $\varphi(A_1) \le \varphi(A_2), \quad A_1 \subset A_2$

(iv) $\varphi\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \varphi(A_i)$ for any countable collection of sets $\{A_i\}$ in $X$.

(iii) says $\varphi$ is <u>monotone</u>

(iv) says that $\varphi$ is countably subadditive

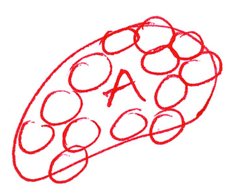
Ex: Let $x_0 \in X$, define

$$\varphi(A) = \begin{cases} 0 & \text{if } x_0 \notin A \\ 1 & \text{if } x_0 \in A \end{cases}$$

$\varphi$ is called the <u>Dirac measure</u> concentrated at $x_0$. Note that $\varphi$ is an outer measure

Ex: $X$ metric space, fix $\varepsilon > 0$.
Define; for each $A \subset X$:

$\varphi(A) =$ Smallest number of balls of radius $\varepsilon$ that cover $A$.



Let

$\mathcal{P}(X) = \{$Collection of all subsets of $X\}$.

$\mathcal{P}(X)$ is the domain of $\varphi$.

We need to have additive properties of $\varphi$. We would like to have

$$\varphi(A \cup B) = \varphi(A) + \varphi(B), \quad A \cap B = \emptyset.$$

In general this is not true.

<u>Def</u>: Let $\varphi$ be an outer measure on a set $X$. A set $E \subset X$ is called $\varphi$-measurable if

$$\varphi(A) = \varphi(A \cap E) + \varphi(A \setminus E), \quad \forall A \subset X.$$

Note: We have

$$\underset{\varphi(A)}{\underset{\shortparallel}{\varphi\left[(A \cap E) \cup (A \cap E^c)\right]}} \le \varphi(A \cap E) + \varphi(A \cap E^c)$$

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Thus, to check that $E$ is $\varphi$-measurable, we only need to check:

$$\varphi(A) \geq \varphi(A \cap E) + \varphi(A \setminus E)$$

<u>Lemma</u>: A set $E \subset X$ is $\varphi$-measurable if and only if:

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q)$$

for any sets $P, Q$, $P \subset E$, $Q \subset E^c$

<u>Proof</u>

$\Leftarrow$ Let $E \subset X$

Let $A \subset X$.

Let $P := A \cap E \subset E$

$Q := A \setminus E \subset E^c$

We have:

$$\begin{aligned} \varphi(P \cup Q) &= \varphi(P) + \varphi(Q) \\ \shortparallel \quad & \\ \varphi(A) &= \varphi(A \cap E) + \varphi(A \setminus E) \end{aligned}$$

Thus, $E$ is measurable

$\Rightarrow$ Assume $E$ is measurable.

Let $P, Q$, $P \subset E$, $Q \subset \tilde{E}$

Then:

$$\varphi(P \cup Q) = \varphi((P \cup Q) \cap E) + \varphi((P \cup Q) \setminus E)$$

$$= \varphi((P \cap E) \cup (Q \cap E)) + \varphi((P \cap E^c) \cup (E^c \cap Q))$$

$$= \varphi(P \cup \emptyset) + \varphi(\emptyset \cup Q)$$

$$= \varphi(P) + \varphi(Q).$$

Ex: Another example of an outer measure.

$$\varphi(A) = \begin{cases} 0 & \text{if } \operatorname{Card} A \leq \aleph_0 \\ 1 & \text{if } \operatorname{Card} A > \aleph_0 \end{cases}$$

$\varphi$ is an outer measure.