Lesson 11.

Thm: For each nonnegative number s, H^s is a Caratheodory outer measure on $\mathbb{R}^n$

Proof: The first three properties are clear.

Let $\{A_i\}$ be a sequence of sets in $\mathbb{R}^n$. Let $\{E_{i,j}\}$ such that:

$$A_i \subset \bigcup_{j=1}^{\infty} E_{i,j}, \quad \operatorname{diam} E_{i,j} \leq \varepsilon$$

$$\sum_{j=1}^{\infty} \alpha(s) 2^{-s} (\operatorname{diam} E_{i,j})^s < H^s_\varepsilon(A_i) + \frac{\varepsilon}{2^i}.$$

Notice:

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i,j} E_{i,j}.$$

$$\Rightarrow H^s_\varepsilon\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha(s) 2^{-s} (\operatorname{diam} E_{i,j})^s$$

$$\leq \sum_{i=1}^{\infty} \left[H^s_\varepsilon(A_i) + \frac{\varepsilon}{2^i} \right]$$

$$\leq \sum_{i=1}^{\infty} H^s(A_i) + \varepsilon; \text{ since, } \forall i: H^s_\varepsilon(A_i) \leq H^s(A_i)$$

Letting $\varepsilon \to 0$:

$$H^s\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} H^s(A_i),$$

which is the subadditivity.

We now show that H^s is a <u>Caratheodory</u> outer measure.

Let $A, B \subset \mathbb{R}^n$, $d(A,B) > 0$.
Let $\{E_i\}$ be a covering of $A \cup B$ with

$$\operatorname{diam} E_i \le \varepsilon, \quad \varepsilon < d(A,B).$$

Let $\mathcal{A}$ be the collection of those E_i that intersect A, and $\mathcal{B}$ those that intersect B. Then

$$\sum_{i=1}^{\infty} \alpha(s) 2^{-s} (\operatorname{diam} E_i)^s = \sum_{E \in \mathcal{A}} \alpha(s) 2^{-s} (\operatorname{diam} E)^s$$

$$+ \sum_{E \in \mathcal{B}} \alpha(s) 2^{-s} (\operatorname{diam} E)^s$$

$$\ge H^s_\varepsilon(A) + H^s_\varepsilon(B).$$

Taking the infimum over all such coverings $\{E_i\}$,

$$H^s_\varepsilon(A \cup B) \ge H^s_\varepsilon(A) + H^s_\varepsilon(B).$$

Letting $\varepsilon \to 0$ we obtain:

$$H^s(A \cup B) \geq H^s(A) + H^s(B)$$

But subadditivity gives:

$$H^s(A \cup B) \leq H^s(A) + H^s(B)$$

$$\therefore \quad H^s(A \cup B) = H^s(A) + H^s(B)$$

Remark; H^s is a Borel outer measure in $\mathbb{R}^n$, since all closed sets are measurable, and hence all Borel sets are measurable.

<u>Thm</u>: For each $A \subset \mathbb{R}^n$, there exists a Borel set $B \supset A$ such that

$$H^s(B) = H^s(A).$$

That is, H^s is a Borel regular outer measure.

<u>Proof</u>:
The sets $\{E_i\}$ in the definition of Hausdorff measure can be taken as closed sets.

Suppose $A \subset \mathbb{R}^n$ with $H^s(A) < \infty$.

$$\therefore \; H^s_\varepsilon(A) < \infty.$$

Let $\varepsilon_j \to 0$, and $\{E_{i,j}\}$ closed sets with:

$$\operatorname{diam} E_{i,j} \le \varepsilon_j$$

$$A \subset \bigcup_{i=1}^{\infty} E_{i,j}$$

$$\sum_{i=1}^{\infty} \alpha(s) 2^{-s} (\operatorname{diam} E_{i,j})^s \le H^s_{\varepsilon_j}(A) + \varepsilon_j$$

Set

$$\boxed{B := \bigcap_{j=1}^{\infty}\left[\bigcup_{i=1}^{\infty} E_{i,j}\right]}$$

B is Borel since each $E_{i,j}$ is Borel ($E_{i,j}$ is closed).

Note:

$$B \subset \bigcup_{i=1}^{\infty} E_{i,j}, \quad \forall j$$

$$\therefore \quad H^s_{\varepsilon_j}(B) \leq \sum_{i=1}^{\infty} \alpha(s)\, 2^{-s} (\operatorname{diam} E_{i,j})^s$$

$$\therefore \quad H^s_{\varepsilon_j}(B) \leq H^s_{\varepsilon_j}(A) + \varepsilon_j$$

Letting $\varepsilon_j \to 0$ gives:

$$\underline{H^s(B) \leq H^s(A)}$$

Since $A \subset \bigcup_{i=1}^{\infty} E_{i,j} \ \forall j \Rightarrow A \subset B$.

$$\therefore \quad \underline{H^s(A) \leq H^s(B)}$$

We conclude: $\boxed{H^s(A) = H^s(B)}$

In the homework, you will improve this result to:

<u>Ex</u>: There is a G_δ set G, $G \supset A$ such that:

$$H^s(G) = H^s(A)$$

<u>Thm</u>: Suppose $A \subset \mathbb{R}^n$, $0 \leq s < t < \infty$.

Then:

(i) $H^s(A) < \infty \implies H^t(A) = 0$

(ii) $H^t(A) > 0 \implies H^s(A) = \infty$.

Proof:

(i) Let $\varepsilon > 0$.

Let $\{E_i\}$ be a covering of A, $\operatorname{diam} E_i < \varepsilon$ such that:

$$\sum_{i=1}^{\infty} \alpha(s) 2^{-s} (\operatorname{diam} E_i)^s \leq H^s_\varepsilon(A) + 1$$
$$\leq H^s(A) + 1$$

Then:

$$H^t_\varepsilon(A) \le \sum_{i=1}^{\infty} \alpha(t)\, 2^{-t} (\operatorname{diam} E_i)^t$$

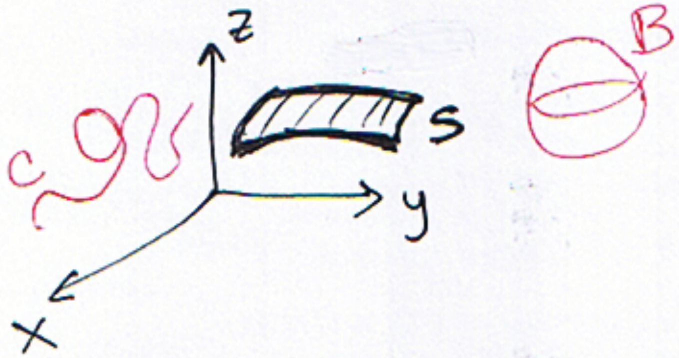
$$= \frac{\alpha(t)}{\alpha(s)}\, 2^{s-t} \underbrace{\sum_{i=1}^{\infty} \alpha(s)\, 2^{-s} (\operatorname{diam} E_i)^s}\, (\operatorname{diam} E_i)^{t-s}$$

$$\le \frac{\alpha(t)}{\alpha(s)}\, 2^{s-t}\, \varepsilon^{t-s} \left[H^s(A) + 1\right]$$

Let $\varepsilon \to 0$ we conclude:

$$H^t(A) = 0 \quad \blacksquare$$

Ex:



$n=3$

- $H^1(S) = \infty$, $0 < H^2(S) < \infty$ (Area of S), $H^3(S) = 0$.
  $H^3 = \lambda_3$. (see exercise 4.38).
  For every $0 \le s \le 3$, H^s is a measure on $\mathbb{R}^3$
- $H^3(B) = \lambda_3(B)$, $H^1(B) = H^2(B) = \infty$
- $H^1(C) =$ length of C, $H^2(C) = H^3(C) = 0$.