Lesson 12

Hausdorff dimension of Cantor Sets.

Recall the definition of Hausdorff measure.

Fix n. For $0 \le s \le n$, define, for each $A \subset \mathbb{R}^n$

$$H^s_\varepsilon(A) = \inf\left\{ \sum_{i=1}^{\infty} \alpha(s) 2^{-s} (\operatorname{diam} E_i)^s,\ A \subset \bigcup_{i=1}^{\infty} E_i,\ \operatorname{diam} E_i < \varepsilon \right\}$$

where:

$$\alpha(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}, \quad \Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\,dx$$

$$H^s(A) = \lim_{\varepsilon \to 0} H^s_\varepsilon(A)$$

- For each $0 \le s \le n$, H^s is a Carathéodory outer measure.

When s is a positive integer, it turns out that $\alpha(s)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^s$. For example, consider $n=3$. In this case

$$\alpha(3) = \frac{\pi^{3/2}}{\Gamma(\frac{3}{2}+1)} = \frac{\pi^{3/2}}{\Gamma(5/2)} = \frac{\pi^{3/2}}{\frac{3}{4}\sqrt{\pi}} = \frac{4}{3}\pi.$$

Note that

$$\alpha(3)\, 2^{-3} (\text{diam } B(x,r))^3 = \frac{4}{3}\pi r^3$$

In fact, it can be shown that:

$$H^n(B(x,r)) = \lambda_n(B(x,r)).$$

Actually, both measures coincide:

$$H^n \equiv \lambda_n \quad \text{(see exercise 4.38)}$$

- When $s=0$, $\alpha(0)\, 2^{-0} (\text{diam } E)^0 = 1$, for any set E. Thus H^0 is a counting measure on $\mathbb{R}^n$. Then, if $\{x_1, \dots, x_p\} \subset \mathbb{R}^n$, $H^0(\{x_1, \dots, x_p\}) = p$. Also, $H^0(\mathbb{Q}_n) = \infty$, where $\mathbb{Q}_n = \{(q_1, \dots, q_n) : q_i \in \mathbb{Q}\}$.
- $H^s \equiv 0$, for all $s > n$ (see exercise 4.40).

Hausdorff dimension:

Def: The Hausdorff dimension of an arbitrary set $A \subset \mathbb{R}^n$ is the number $0 \le \delta_A \le n$ such that

$$\delta_A := \inf\{t : H^t(A) = 0\}.$$

In other words, the Hausdorff dimension δ_A is that unique number such that:

$$s < \delta_A \Rightarrow H^s(A) = \infty$$
$$t > \delta_A \Rightarrow H^t(A) = 0.$$

- If $s = \delta_A$ then one of the following three possibilities has to occur:

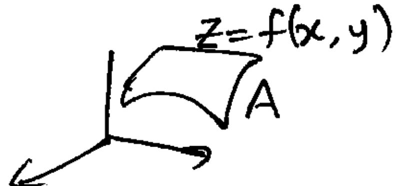
(a) $H^s(A) = 0$

(b) $H^s(A) = \infty$

(c) $0 < H^s(A) < \infty$.

On the other hand, if $0 < H^s(A) < \infty \Rightarrow \delta_A = s$.

Ex:



Since $0 < H^2(A) < \infty$, then $\delta_A = 2$

Ex: A countable set $A \subset \mathbb{R}^n$ has Hausdorff dimension 0. (Problem 4.44)

Indeed, $H^0(A) = \infty$ and $H^t(A) = 0$, $t > 0$.

The oppositive is not true: Hausdorff dimension $0 \not\Rightarrow A$ is countable. (see next example).

<u>Ex.</u> There exists a cantor set $C(S) \subset [0,1]$ such that $C(S)$ is uncountable and its Hausdorff dimension is zero. (see problem 4.45).

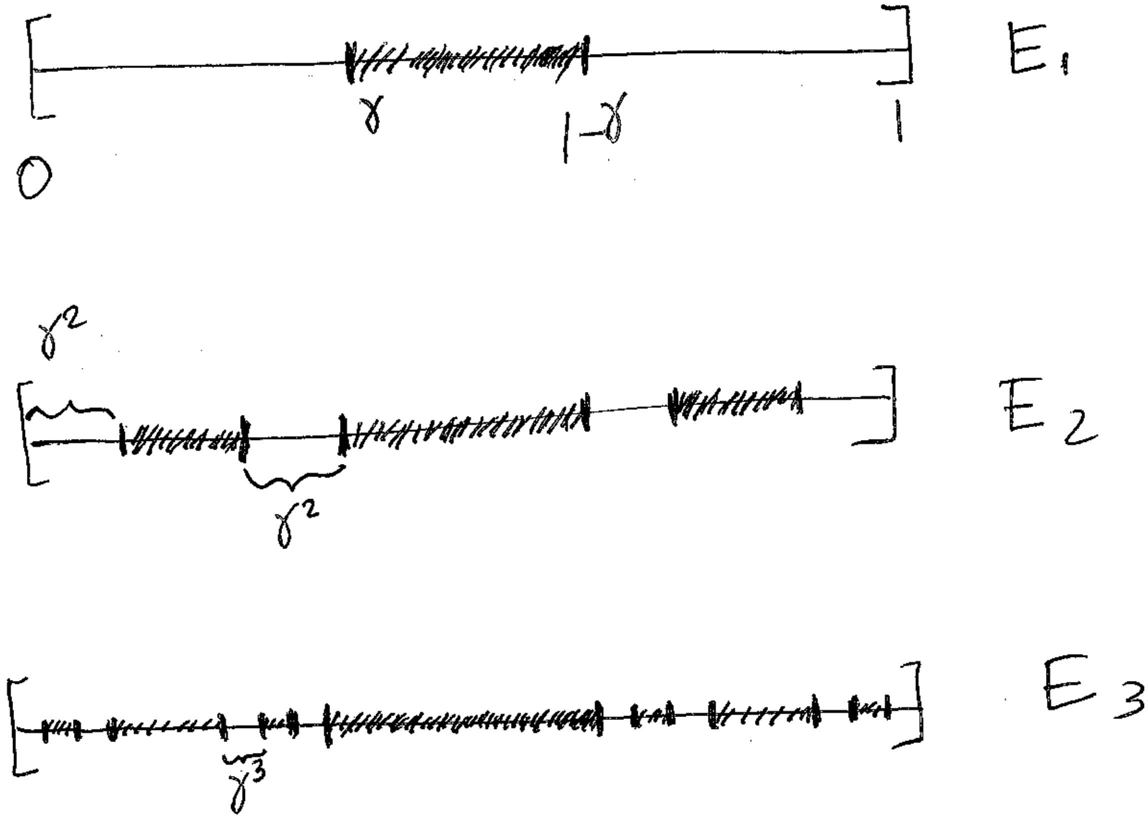
<u>Ex</u>: There exists a Cantor set $C(S) \subset [0,1]$ of dimension 1. (problem 4.45), which shows that there are sets other than intervals in $\mathbb{R}$ that have dimension 1. In general, there are examples of extremely complicated Cantor-like subsets A of $\mathbb{R}^n$ with $0 < H^k(A) < \infty$. Thus, even though $\delta_A = k$, A need not resemble a "k-dimensional surface" in any usual sense.

<u>Ex</u>: Recall, if $s < t$: $\left. \begin{array}{l} H^s(A) < \infty \Rightarrow H^t(A) = 0 \\ \text{or} \quad H^t(A) > 0 \Rightarrow H^s(A) = \infty . \end{array} \right\}$ $(*)$

Let $U \subset \mathbb{R}^n$ an open set. If $H^n(U) = \lambda_n(U) = \infty$, then by $(*)$ $H^s(U) = \infty$ $\forall s < n$; and since $H^s(U) = 0$ $\forall s > n$. Hence $\delta_U = n$. If $0 < H^n(U) = \lambda_n(U) < \infty$ then $(*)$ implies again that $\delta_U = n$.

Ex: General Cantor set

Let $0 < \gamma < \frac{1}{2}$



$\vdots$

E_k has 2^k intervals, each of length γ^k.

Define:

$$E_K := \bigcup_{j=1}^{2^K} I_{K,j}$$

Define:

$$C(\gamma) := \bigcap_{K=1}^{\infty} E_K .$$

You will show in hw that

$$H^1 = \lambda \text{ on } \mathbb{R} .$$

Each E_K is measurable. Thus:

$$\begin{aligned}
\lambda(C(\gamma)) &= \lim_{K\to\infty} \lambda(E_K) \\
&= \lim_{K\to\infty} 2^K \gamma^K \\
&= \lim_{K\to\infty} (2\gamma)^K , \quad 2\gamma < 1 \\
&= 0 .
\end{aligned}$$

Thus:

$$H^1(C(\gamma)) = \lambda(C(\gamma)) = 0$$

<u>Claim</u>: $\delta_{C(\gamma)} = \dfrac{\log 2}{\log(1/\gamma)}$ $(*)$

<u>Note</u>: If $\gamma = \frac{1}{3}$, $C(\gamma)$ is the original Cantor set constructed in class and thus:

$$\delta_{C(1/3)} = \frac{\log 2}{\log 3} = 0.63$$

In order to see $(*)$ we compute:

$$H^s_{2\beta^k}(C(\gamma)) \le \sum_{j=1}^{2^k} \alpha(s) 2^{-s} (\operatorname{diam} I_{k,j})^s$$

$$\left(\text{since } C(\gamma) \subset \bigcup_{j=1}^{2^k} I_{k;j}\right)$$

$$= \sum_{j=1}^{2^k} \alpha(s) 2^{-s} (\beta^k)^s = \alpha(s) 2^{-s} 2^k \beta^{ks}$$

$$\therefore \quad H^s_{2\beta^k}(C(\gamma)) \le \alpha(s) 2^{-s} (2\beta^s)^k, \ \forall k$$

Let s be such that (12.8)

$$2\beta^s = 1 \; ; \; \text{i.e}$$

$$s = \frac{\log 2}{\log(1/\beta)}$$

With this s:

$$H^s(C(\gamma)) = \lim_{K\to\infty} H^s_{2\beta^K}(C(\gamma))$$

$$\leq \alpha(s) 2^{-s} < \infty .$$

It can also be shown that

$$H^s(C(\gamma)) > 0$$

Therefore:

$$0 < H^s(C(\gamma)) < \infty . \qquad (**)$$

But $(**)$ implies.

$$\delta_{C(\gamma)} = s = \frac{\log 2}{\log(1/\beta)} .$$