Lesson 13

4.9

Measures on Abstract spaces.

<u>Def</u>: Let X be a set and $\mathcal{M}$ a σ-algebra of subsets of X. A <u>measure</u> on $\mathcal{M}$ is a function

$\mu : \mathcal{M} \to [0, \infty]$ s.t.

(i) $\mu(\phi) = 0$

(ii) If $\{E_i\}$ is a sequence of disjoint sets in $\mathcal{M}$, then:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

$(X, \mathcal{M}, \mu)$ is called a <u>measure space</u> and the sets in $\mathcal{M}$ are called <u>measurable sets</u>.

In case $\mathcal{M}$ constitutes the family of Borel sets in a metric space X, μ is called a <u>Borel measure</u>.

A measure μ is said to be finite if $\mu(X) < \infty$.

A measure μ is said to be σ-finite if

$$X = \bigcup_{i=1}^{\infty} E_i,$$

for some E_i, $\mu(E_i) < \infty$.

<u>Ex</u>: $(X, \mathcal{M}, \varphi)$, where φ is an outer measure on an abstract space X and $\mathcal{M}$ is the family of φ-measurable sets.

<u>Ex</u>: $(X, \mathcal{M}, \mu)$ where $\mathcal{M}$ is the family of all subsets of an arbitrary space X and

$\mu(E)$ = number (possibly infinity) of points in E.

<u>Thm</u>: Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_i\}$ is a sequence of sets in $\mathcal{M}$. Then

(i) $E_1 \subset E_2 \Rightarrow \mu(E_1) \le \mu(E_2)$

(ii) $E_1 \subset E_2$, $\mu(E_1) < \infty \Rightarrow$

$$\mu(E_2 \setminus E_1) = \mu(E_2) - \mu(E_1)$$

(iii) $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu(E_i)$

(iv) If $E_i \subset E_{i+1} \Rightarrow$

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} \mu(E_i)$$

(v) If $\mu(E_{i_0}) < \infty$, some $i_0 \Rightarrow \mu\left(\bigcap^{\infty} E_i\right) = \lim_{i \to \infty} \mu(E_i)$

The following property is characteristic to an outer measure φ but it is not enjoyed by abstract measures in general:

"If $\varphi(E)=0$, then E is φ-measurable and if $A \subset E$ then A is φ-measurable (since $\varphi(A) \leq \varphi(E)=0$)".

A measure μ with the property that all subsets of sets of μ-measure zero are measurable, is said to be complete and $(X, \mathcal{M}, \mu)$ is called a complete measure ~~measure~~ space.

Not all measures are complete but every measure can be completed using the following theorem.

<u>Thm</u>: Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Define

$$\overline{\mathcal{M}} = \{ A \cup N : A \in \mathcal{M},\ N \subset B \text{ for some } B \in \mathcal{M} \text{ such that } \mu(B) = 0 \}$$

and define $\bar{\mu}$ on $\overline{\mathcal{M}}$ by

$$\bar{\mu}(A \cup N) = \mu(A).$$

Then

(a) $\overline{\mathcal{M}}$ is a σ-algebra

(b) $\bar{\mu}$ is a complete measure on $\overline{\mathcal{M}}$ and $(X, \overline{\mathcal{M}}, \bar{\mu})$ is a complete measure space

(d) $\bar{\mu}$ is the only complete measure on $\overline{\mathcal{M}}$ that is an extension of μ.

<u>Proof</u>: $\overline{\mathcal{M}}$ is closed under countable unions, since

$$\bigcup_{i=1}^{\infty} (A_i \cup N_i) = \underbrace{\left(\bigcup_{i=1}^{\infty} A_i \right)}_{\in \mathcal{M}} \cup \left(\bigcup_{i=1}^{\infty} N_i \right)$$

and $\bigcup_{i=1}^{\infty} N_i \subset \bigcup_{i=1}^{\infty} B_i$ with $\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = 0$

Let $A \cup N \in \overline{\mathcal{M}}$, $N \subset B$, $\mu(B)=0$ (3.6)

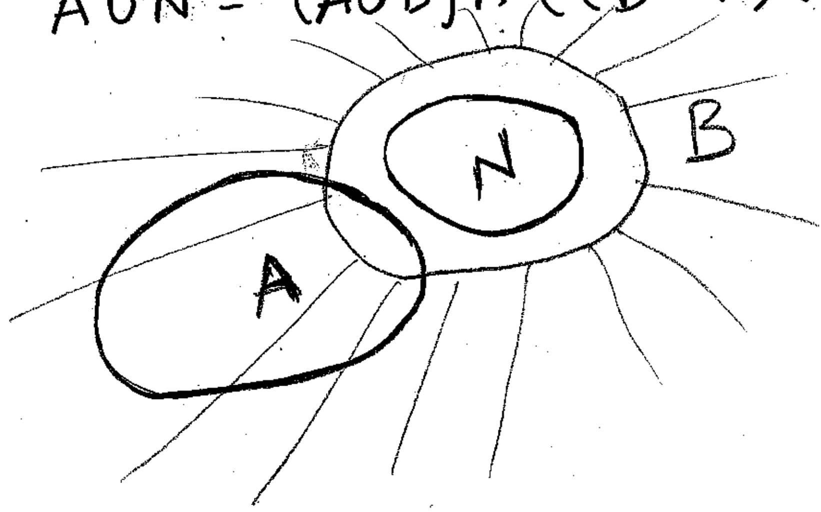
We can assume $A \cap N = \emptyset$ since otherwise we can work with

$$A \cup (N \setminus A), \quad N \setminus A \subset B \setminus A,$$

$$B \setminus A \in \mathcal{M}, \quad \mu(B \setminus A) \le \mu(B) = 0 .$$

With $A \cap N = \emptyset$ we write.

$$A \cup N = (A \cup B) \cap \left((B^c \cup N) \cup (A \cap B) \right)$$



$$\begin{aligned}
(A \cup N)^c &= (A \cup B)^c \cup \left[(B^c \cup N) \cup (A \cap B) \right]^c \\
&= (A \cup B)^c \cup \left[(B^c \cup N)^c \cap (A \cap B)^c \right] \\
&= (A \cup B)^c \cup \left[(B \cap N^c) \cap (A \cap B)^c \right] \\
&= (A \cup B)^c \cup \left[(B \setminus N) \setminus (A \cap B) \right]
\end{aligned}$$

$\Rightarrow$

$(A \cup B)^c \in \mathcal{M}$

$(B \setminus N) \setminus (A \cap B) \subset B, \quad \mu(B) = 0$

$\Rightarrow \overline{\mathcal{M}}$ is a σ-algebra.

We now show that $\bar{\mu}$ is well defined.

Suppose $\quad A_1 \cup N_1 = A_2 \cup N_2$

$N_1 \subset B_1, \quad N_2 \subset B_2, \quad \mu(B_1) = \mu(B_2) = 0$

$$\begin{aligned}
\bar{\mu}(A_1 \cup N_1) = \mu(A_1) &\le \mu(A_2 \cup B_2), \quad A_1 \subset A_2 \cup B_2 \\
&\le \mu(A_2) + \mu(B_2) \\
&= \mu(A_2) \\
&= \bar{\mu}(A_2 \cup N_2)
\end{aligned}$$

$\Rightarrow \quad \bar{\mu}(A_1 \cup N_1) \le \bar{\mu}(A_2 \cup N_2)$

In the same way:

$\bar{\mu}(A_2 \cup N_2) \le \bar{\mu}(A_1 \cup N_1)$.

$\bar{\mu}$ is complete since:

$E \subset A \cup N, \ \underbrace{\bar{\mu}(A \cup N)}_{\overset{\shortparallel}{\mu(A)}} = 0 \Rightarrow \bar{\mu}(E) = \bar{\mu}(\emptyset \cup E) = \mu(\emptyset) = 0.$

$(N \subset B, \mu(B) = 0 \Rightarrow E \subset A \cup B, \mu(A \cup B) = 0)$

$\bar{\mu}$ is an extension of μ.

Let $A \in \mathcal{M}$

$\Rightarrow \bar{\mu}(A) = \bar{\mu}(A \cup \emptyset) = \mu(A)$.