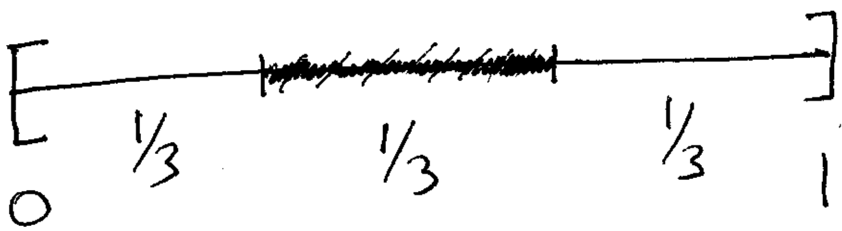
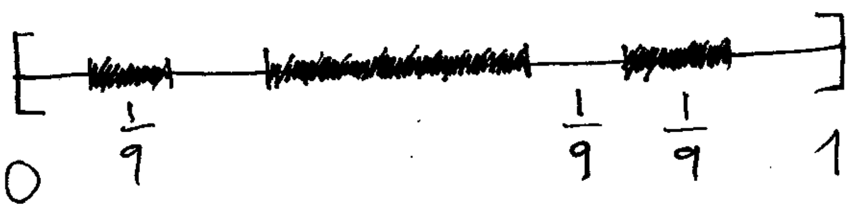
Ex: The Cantor–Lebesgue Function.

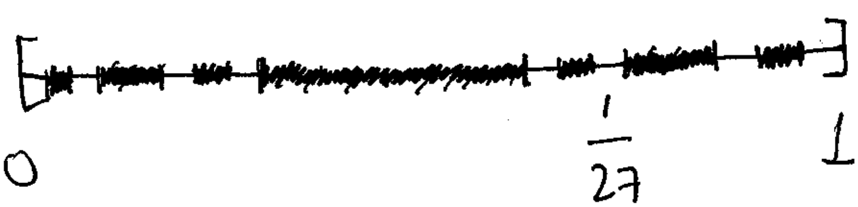
We start with the Cantor set:



E_1



E_2



E_3

$\vdots$

$$C = \bigcap_{k=1}^{\infty} E_k$$

E_k has 2^k intervals of size $\frac{1}{3^k}$.

Let

$$G_K = [0,1] \setminus E_K$$

G_K has $2^K - 1$ intervals, $I_1^K, I_2^K, \dots, I_{2^K-1}^K$

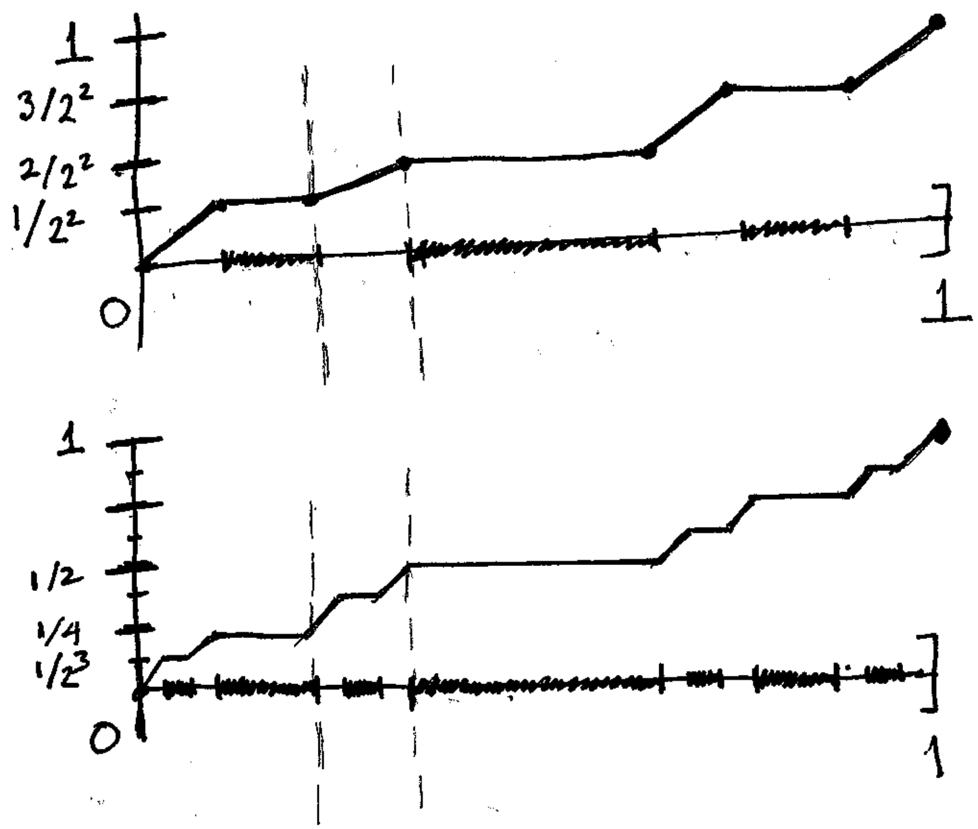
Define:

$$f_K : [0,1] \to \mathbb{R}.$$

$$f_K(0) = 0, \qquad f_K(1) = 1$$

$$f_K(x) = \frac{i}{2^K}, \quad \text{if } x \in I_i^K,$$

$$i \in \{1, 2, \dots, 2^K - 1\}.$$



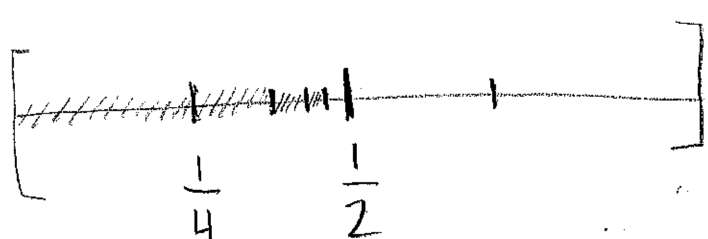
Let $x \in [0,1]$.

We note that

$$|f_K(x) - f_{K+1}(x)| < \frac{1}{2^K}.$$

Also, for every $m > 0$, and every $x \in [0,1]$

$$\begin{aligned}
&|f_K(x) - f_{K+m}(x)| \\
&\leq |f_K(x) - f_{K+1}(x)| + |f_{K+1}(x) - f_{K+2}(x)| \\
&\quad + |f_{K+2}(x) - f_{K+3}(x)| + \cdots + \\
&\quad \cdots + |f_{K+m-1}(x) - f_{K+m}(x)| \\
&< \frac{1}{2^K} + \frac{1}{2^{K+1}} + \frac{1}{2^{K+2}} + \cdots + \frac{1}{2^{K+m-1}}
\end{aligned}$$



$K = 2$

$$< \sum_{i=K}^{\infty} \frac{1}{2^i} = \frac{1}{2^K}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$$

$$= \frac{1}{2^K} \cdot \frac{1}{1-\frac{1}{2}} = \frac{2}{2^K} = \frac{1}{2^{K-1}}$$

Thus, given any $\varepsilon > 0$, we can find N s.t

$$\frac{1}{2^{N-1}} < \varepsilon.$$

Thus; for every $k, j \geq N$, $k < j$:

$$|f_k(x) - f_j(x)| < \frac{1}{2^{k-1}} < \frac{1}{2^{N-1}} < \varepsilon$$

$$\Rightarrow |f_k(x) - f_j(x)| < \varepsilon \quad \forall k, j \geq N, \; k < j, \; \forall x \in [0,1]$$

$\Rightarrow \{f_k\}$ is uniformly Cauchy in $[0,1]$

Thus $\{f_k\}$ converges uniformly to a function $f: [0,1) \to \mathbb{R}$.

Since each f_k is continuous, the uniform convergence implies:

f is continuous on $[0,1]$

f is called the Cantor-Lebesgue-function. Note:

(i) f is constant on each interval in $[0,1]\setminus C$.

(ii) f is nondreasing:

Let $x<y$, $x,y\in[0,1]$

$$\Rightarrow \quad f_K(x) \le f_K(y) \qquad \forall K$$

$$\Rightarrow \lim_{K\to\infty} f_K(x) \le \lim_{K\to\infty} f_K(y)$$

$$\Rightarrow \quad f(x) \le f(y)$$

(iii) $f([0,1]) = [0,1]$

(iv) $f(C) = [0,1]$

In order to see (iv), we compute

$$f([0,1]) = \underbrace{f(C) \cup f([0,1]\setminus C)}_{B}.$$

Since $f([0,1]\setminus C)$ is countable (taking the values $\frac{i}{2^K}$) then:

$$\lambda\left[f([0,1]\setminus C)\right] = 0$$

$$\Rightarrow \lambda(B) \le \lambda(f(C)) + \lambda(f([0,1]\setminus C))$$

$\overset{\shortparallel}{}$

And
$$\lambda(f([0,1]))=1$$
$\therefore$ $$\lambda(f(C)) \geq 1$$

Since $f(C) \subset [0,1]$
$$\Rightarrow \lambda(f(C)) \leq \lambda([0,1]) = 1$$

$\therefore$ $$\lambda(f(C)) = 1.$$

Since C is compact and f is continuous, we have
$$f(C) \text{ is compact}$$
$$\Rightarrow f(C) \text{ is closed and bounded.}$$

We proceed by contradiction and assume that
$$(0,1) \setminus f(C) \neq \emptyset$$

Then, since $(0,1)\setminus f(C)$ is open it follows that there exists $x \in (0,1)\setminus f(C)$ and an open interval (a,b), $x \in (a,b)$, such that:

$$x \in (a,b) \subset [0,1) \setminus f(C)$$

Therefore we have, since $(a,b) \cap f(C) = \emptyset$:

$$\lambda(f(C) \cup (a,b)) = \lambda(f(C)) + \lambda(a,b) = 1 + (b-a). \quad (1)$$

On the other hand, $f(C) \cup (a,b) \subset [0,1]$ yields:

$$\lambda(f(C) \cup (a,b)) \leq \lambda([0,1]) = 1,$$

which contradicts (1).

We have shown that $f(C) = (0,1)$, and since $f(0)=0$, $f(1)=1$, $\{0,1\} \subset C$, we conclude that $f(C) = [0,1]$.

The composition of Lebesgue measurable functions need not be Lebesgue measurable.

Def: $h(x) = f(x) + x$

$\Rightarrow$ h is strictly increasing

$\Rightarrow$ $h : [0,1] \to [0,2]$ is 1-1 and on-to.

Since f is constant on the intervals in $[0,1]\setminus C$, and

Since $\lambda([0,1]\setminus C)=1$, we have that:

$$\lambda(h([0,1]\setminus C)) = 1.$$

$$\Rightarrow \quad \lambda(h(C)) = 1$$

Let N be a non-Lebesgue measurable subset of $h(C)$. Define:

$$A := h^{-1}(N)$$

$A \subset C$ and thus $\lambda(A) \leq \lambda(C) = 0$.

A is measurable (λ is a complete measure).

$\therefore$ h maps a measurable set onto a nonmeasurable set.

$$\begin{tikzcd}
[0,1] \arrow[r, shift left, "h"] & {[0,2]} \arrow[l, shift left, "g"] \\
A \subset C \arrow[r, leftrightarrow] & N \subset h(C)
\end{tikzcd}$$

Let $F = h^{-1} : [0,2] \to [0,1]$

Recall:

<u>Thm</u>: $f : X \to Y$ 1-1 on-to, continuous,

X compact metric space

Y metric space

$\Rightarrow f^{-1} : Y \to X$ is continuous.

$\therefore$ F is continuous

$\therefore$ F is Lebesgue measurable

Note that

$$F^{-1}(A) = N$$

A is not Borel. (Otherwise N would be measurable since g is Lebesgue measurable).

<u>Note</u>: A Lebesgue measurable

$\iff$ χ_A is Lebesgue measurable.

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Note that.

$$\chi_A \circ h^{-1} = \chi_N.$$

χ_A Lebesgue measurable
h^{-1} Lebesgue measurable
χ_N Non-Lebesgue measurable.

<u>Conclusion</u>: Composition of Lebesgue-measurable is not Lebesgue-measurable.