Lecture 17

Recall the Cantor–Lebesgue Function:

- $f: [0,1] \to [0,1]$
- f continuous, non-decreasing, on-to
- $f(C) = [0,1]$

f maps a set of measure 0 onto a set of positive measure (recall that if f is a Lipschitz function, then f maps sets of measure zero to sets of measure zero).

Defining: $h(x) = f(x) + x$

we have

- $h: [0,1] \to [0,2]$ is 1-1 and on-to, continuous.
- $F := h^{-1}$ is continuous
- $\lambda(h(C)) = 1$

17.2

$$[0,1] \underset{F}{\overset{h}{\rightleftarrows}} [0,2]$$

$$A \subset C \underset{F}{\longleftrightarrow} N \subset h(C)$$

$$A \longleftrightarrow N$$

Since F is continuous, then F is Borel measurable (and hence Lebesgue measurable).

Recall the definitions ($\mathcal{B}$ denotes the σ-algebra of Borel sets):

$$f : (\mathbb{R}^n, \mathcal{M}) \longrightarrow (\overline{\mathbb{R}}, \mathcal{B})$$

is Lebesgue measurable if $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets and $f^{-1}(B) \in \mathcal{M}$ for every Borel set $B \in \mathcal{B}$.

$$f : (\mathbb{R}^n, \mathcal{B}) \longrightarrow (\overline{\mathbb{R}}, \mathcal{B})$$

is Borel measurable if $f^{-1}(B)$ is Borel for every Borel set $B \in \mathcal{B}$.

<u>Note</u>: A is not Borel, since otherwise we would have $F^{-1}(A) \in \mathcal{B}$. But $F^{-1}(A) = N \notin \mathcal{B}$.

<u>Note</u>: A is measurable because it has measure zero. Indeed:

$$\lambda(A) \leq \lambda(C) = 0.$$

- h is a continuous function that maps a Lebesgue measurable set onto a non-measurable set.

> The composition of Lebesgue measurable functions need not be Lebesgue measurable

Let

$$g := \chi_A$$

- Since $g^{-1}(1) = A$ the g is <u>NOT</u> Borel measurable. However, g is Lebesgue measurable. So we have:

$$f \text{ Borel measurable} \implies f \text{ Lebesgue measurable}$$

$$f \text{ Lebesgue measurable} \not\Longrightarrow f \text{ Borel measurable}$$

Consider the composition

$$g\circ F = \underbrace{\chi_A}_{\text{Lebesgue measurable}} \circ \underbrace{h^{-1}}_{\text{Lebesgue measurable}} = \underbrace{\chi_N}_{\text{Non-Lebesgue measurable}}$$

However, we have

<u>Thm 1</u>: Suppose $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is Lebesgue measurable and $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is Borel measurable. Then $g\circ f$ is Lebesgue measurable

Proof: in this case

$$\mathbb{R}^n \xrightarrow{f} \overline{\mathbb{R}} \xrightarrow{g} \overline{\mathbb{R}}$$

$$g\circ f : \mathbb{R}^n \to \overline{\mathbb{R}}$$

If B is a Borel set in $\overline{\mathbb{R}}$ then:

$$(g\circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$$

g Borel measurable $\Rightarrow$ $g^{-1}(B)$ is Borel

f Lebesgue measurable $\Rightarrow$ $f^{-1}(g^{-1}(B))$ is Lebesgue measurable

$\therefore$ $g\circ f$ is Lebesgue-measurable.

Remark: Let $g : \mathbb{R} \to \overline{\mathbb{R}}$ be Borel measurable. We can extend g to $\overline{\mathbb{R}}$ by assigning arbitrary values to ∞ and $-\infty$. The extension $\bar{g} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is Borel measurable.

Corollary: Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be Lebesgue measurable

(i) Let $\varphi(x) = |f(x)|^p$, $0 < p < \infty$, where φ assumes arbitrary extended values on the sets $f^{-1}(\infty)$, and $f^{-1}(-\infty)$. Then φ is Lebesgue measurable

(ii) Let $\varphi(x) = \dfrac{1}{f(x)}$, and let φ assume arbitrary extended values on the sets $f^{-1}(0)$, $f^{-1}(\infty)$ and $f^{-1}(-\infty)$. Then φ is Lebesgue measurable.

Proof.

(i) Define

$g(t) = |t|^p$, $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, $g(\infty) =$ arbitrary value,

$g(-\infty) =$ arbitrary value.

$t \mapsto |t|^p$, $t \in \mathbb{R}$ is continuous $\Rightarrow$ $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is Borel measurable.

$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ Lebesgue measurable

Thm 1 $\Rightarrow$ $g \circ f$ is Lebesgue measurable

(ii) Define:

$g(t) = \frac{1}{t}$, $g(0) =$ arbitrary value,

$g(\infty) =$ arbitrary, $g(-\infty) =$ arbitrary

$t \mapsto \frac{1}{t}$, $t \in \mathbb{R}$ is continuous $\Rightarrow$

$g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is Borel measurable and

$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is Lebesgue measurable

Thm 1 $\Rightarrow$ $g \circ f$ is Lebesgue measurable.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

A measurable set is called a μ-null set if $\mu(N) = 0$.

A property that holds for all $x \in X$ except for those x in some μ-null set is said to hold

μ - almost everywhere

or

μ-a.e.

If it is clear from context that the measure μ is under consideration, we will use the terms:

"null set", and

"almost everywhere".

<u>Thm</u>: Let $(X, \mathcal{M}, \mu)$ be a complete measure space and let $f, g : X \to \overline{\mathbb{R}}$.

If f is measurable and $f = g$ almost everywhere, then g is measurable.

<u>Proof</u>: Let $N = \{x : f(x) = g(x)\}$.

$$\Rightarrow \mu(X \setminus N) = 0$$

Let $a \in \mathbb{R}$

$$\{g > a\} = (\{g > a\} \cap N) \cup (\{g > a\} \cap (X \setminus N))$$
$$= (\{f > a\} \cap N) \cup (\{g > a\} \cap (X \setminus N))$$

$\{f > a\} \in \mathcal{M}$, $N \in \mathcal{M}$, $X \setminus N \in \mathcal{M}$

$\Rightarrow \{f > a\} \cap N \in \mathcal{M}$

Since X is complete $\Rightarrow$

$\{g > a\} \cap (X \setminus N) \in \mathcal{M}$

$\Rightarrow \{g > a\} \in \mathcal{M}$

If we define:

$f \sim g$ if $f = g$ μ-almost everywhere

Then $\sim$ defines an equivalence relation and hence a function can be regarded as an equivalence class of functions.

(If $(X, \mathcal{M}, \mu)$ is complete).

5.2

Limits of Measurable functions.

In this section all functions are $\overline{\mathbb{R}}$-valued.

<u>Def</u>: $(X, \mathcal{M}, \mu)$ measure space

$\{f_i\}$ measurable on X.

The upper and lower envelopes of $\{f_i\}$ are defined:

$$\sup_i f_i(x) = \sup \{f_i(x) : i = 1, 2, \dots\}$$

$$\inf_i f_i(x) = \inf \{f_i(x) : i = 1, 2, \dots\}$$

Def.

$$(\limsup_{i\to\infty} f_i)(x) = \inf_{j\geq 1} \left(\sup_{i\geq j} f_i(x)\right)$$

$$(\liminf_{i\to\infty} f_i)(x) = \sup_{j\geq 1} \left(\inf_{i\geq j} f_i(x)\right)$$

Thm: $\sup\limits_i f_i$, $\inf\limits_i f_i$,

$\limsup\limits_{i\to\infty} f_i$, $\liminf\limits_{i\to\infty} f_i$

are all measurable.

Proof: Let $a \in \mathbb{R}$.

$$\underbrace{\{x : (\sup_i f_i) > a\}}_{A} = \underbrace{\bigcup_{i=1}^{\infty} \{x : f_i(x) > a\}}_{B}$$

Let $x \in A$

$\iff (\sup\limits_i f_i)(x) > a$

$\iff \sup\{f_i(x)\} > a$ $\qquad \alpha = \sup\{f_i(x)\}$ $\quad a$

$\iff f_j(x) > a$, for some j

$\iff x \in \{x : f_j(x) > a\}$, some j

$\iff x \in B$

$$\inf_i f_i(x) = -\sup_i (-f_i(x))$$

<u>Def</u>: Let $\{f_i\}$ be a sequence of measurable functions, such that

$$\lim_{i\to\infty} f_i(x) = f(x)$$

for μ- almost every $x \in X$.

Then we say:

" f_i converges pointwise almost everywhere"

or

" f_i converges pointwise a.e.", to f.

<u>Thm</u>: (Egoroff).

Let $(X, \mathcal{M}, \mu)$ finite measure space

Let $\{f_j\}$ and $\{f\}$ be measurable and finite almost everywhere on X.

Assume $f_j \to f$ pointwise a.e

Then:

$\forall \varepsilon > 0$, $\exists A \in \mathcal{M}$, s.t. $\mu(A^c) < \varepsilon$ and

$f_j \to f$ uniformly on A.