

Lecture 17

Recall the Cantor-Lebesgue Function:

- $f: [0, 1] \rightarrow [0, 1]$
- f continuous, non-decreasing, on-to
- $f(C) = [0, 1]$

f maps a set of measure 0 onto a set of positive measure (recall that if f is a Lipschitz function, then f maps sets of measure zero to sets of measure zero).

Defining: $h(x) = f(x) + x$

we have

- $h: [0, 1] \rightarrow [0, 2]$
is 1-1 and on-to, continuous.
- $F := h^{-1}$ is continuous
- $\lambda(h(C)) = 1$

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$$\begin{array}{ccc} [0, 1] & \xrightarrow{h} & [0, 2] \\ & \xleftarrow{F} & \\ \text{ACC} & \xleftarrow{F} & \text{NCH}(C) \end{array}$$

$$A \longleftrightarrow N$$

Since F is continuous, then F is Borel measurable (and hence Lebesgue measurable).

Recall the definitions (\mathcal{B} denotes the σ -algebra of Borel sets):

$$f: (\mathbb{R}^n, \mathcal{M}) \rightarrow (\bar{\mathbb{R}}, \mathcal{B})$$

is Lebesgue measurable if \mathcal{M} is the σ -algebra of Lebesgue measurable sets and $f^{-1}(B) \in \mathcal{M}$ for every Borel set $B \in \mathcal{B}$.

$$f: (\mathbb{R}^n, \mathcal{B}) \rightarrow (\bar{\mathbb{R}}, \mathcal{B})$$

is Borel measurable if $f^{-1}(B)$ is Borel for every Borel set $B \in \mathcal{B}$.

Note: A is not Borel, since otherwise we would have $F^{-1}(A) \in \mathcal{B}$.
But $F^{-1}(A) = N \notin \mathcal{B}$.

Note: A is measurable because it has measure zero. Indeed:
 $\lambda(A) \leq \lambda(C) = 0$.

- h is a continuous function that maps a Lebesgue measurable set onto a non-measurable set.

The composition of Lebesgue measurable functions need not be Lebesgue measurable

Let

$$g := \chi_A$$

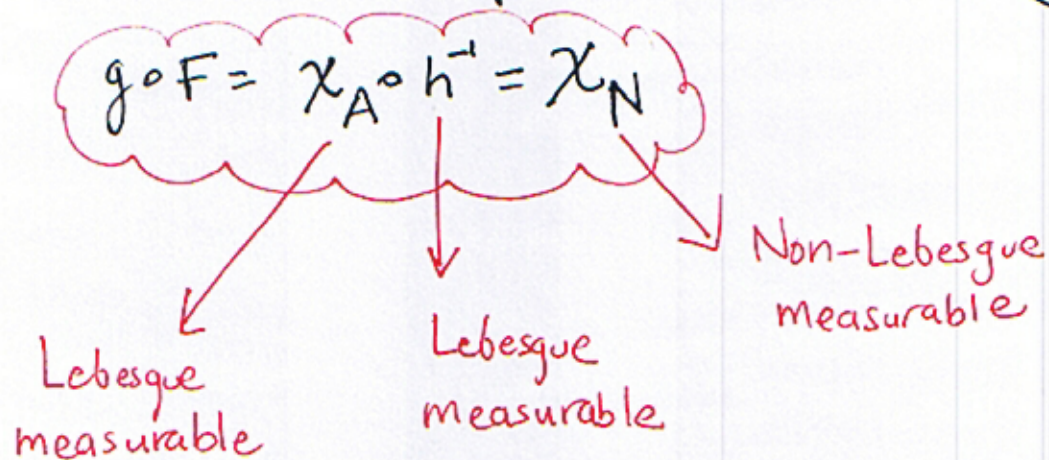
- Since $g^{-1}(1) = A$ the g is NOT Borel measurable. However, g is Lebesgue measurable. So we have:

$$f \text{ Borel measurable} \Rightarrow f \text{ Lebesgue measurable}$$

$$f \text{ Lebesgue measurable} \not\Rightarrow f \text{ Borel measurable}$$

Consider the composition

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However, we have

Thm 1: Suppose $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is Lebesgue measurable and $g: \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ is Borel measurable. Then $g \circ f$ is Lebesgue measurable

Proof: in this case

$$\mathbb{R}^n \xrightarrow{f} \overline{\mathbb{R}} \xrightarrow{g} \overline{\mathbb{R}}$$
$$g \circ f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$$

If B is a Borel set in $\overline{\mathbb{R}}$ then:

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$$

g Borel measurable $\Rightarrow g^{-1}(B)$ is Borel

f Lebesgue measurable $\Rightarrow f^{-1}(g^{-1}(B))$ is Lebesgue measurable

$\therefore g \circ f$ is Lebesgue-measurable.

Remark: Let $g: \mathbb{R} \rightarrow \bar{\mathbb{R}}$ be Borel measurable. We can extend g to $\bar{\mathbb{R}}$ by assigning arbitrary values to ∞ and $-\infty$. The extension $\bar{g}: \bar{\mathbb{R}} \rightarrow \bar{\mathbb{R}}$ is Borel measurable.

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Corollary: Let $f: \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ be Lebesgue measurable

(i) Let $\varphi(x) = |f(x)|^p$, $0 < p < \infty$, where φ assumes arbitrary extended values on the sets $f^{-1}(\infty)$, and $f^{-1}(-\infty)$. Then φ is Lebesgue measurable

(ii) Let $\varphi(x) = \frac{1}{f(x)}$, and let φ assume arbitrary extended values on the sets $f^{-1}(0)$, $f^{-1}(\infty)$ and $f^{-1}(-\infty)$. Then φ is Lebesgue measurable.

Proof.

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(i) Define

$$g(t) = |t|^p, \quad g: \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}, \quad g(\infty) = \text{arbitrary value,}$$

$$g(-\infty) = \text{arbitrary value.}$$

$t \mapsto |t|^p, t \in \mathbb{R}$ is continuous $\Rightarrow g: \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ is Borel measurable.

$$f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}} \quad \text{Lebesgue measurable}$$

Thm 1 $\Rightarrow g \circ f$ is Lebesgue measurable

(ii) Define:

$$g(t) = \frac{1}{t}, \quad g(0) = \text{arbitrary value,}$$

$$g(\infty) = \text{arbitrary,} \quad g(-\infty) = \text{arbitrary}$$

$$t \mapsto \frac{1}{t}, t \in \mathbb{R} \text{ is continuous } \Rightarrow$$

$g: \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ is Borel measurable and $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is Lebesgue measurable

Thm 1 $\Rightarrow g \circ f$ is Lebesgue measurable.

Let (X, \mathcal{M}, μ) be a measure space.

A measurable set is called a μ -null set if $\mu(N) = 0$.

A property that holds for all $x \in X$ except for those x in some μ -null set is said to hold

μ -almost everywhere

or

μ -a.e.

If it is clear from context that the measure μ is under consideration, we will use the terms:

"null set", and
"almost everywhere".

Thm: Let (X, \mathcal{M}, μ) be a complete measure space and let $f, g: X \rightarrow \overline{\mathbb{R}}$.

If f is measurable and $f=g$ almost everywhere, then g is measurable.

Proof: Let $N = \{x: f(x) \neq g(x)\}$.

$$\Rightarrow \mu(X \setminus N) = 0$$

Let $a \in \mathbb{R}$

$$\begin{aligned} \{g > a\} &= (\{g > a\} \cap N) \cup (\{g > a\} \cap (X \setminus N)) \\ &= (\{f > a\} \cap N) \cup (\{g > a\} \cap (X \setminus N)) \end{aligned}$$

$$\{f > a\} \in \mathcal{M}, N \in \mathcal{M}, X \setminus N \in \mathcal{M}$$

$$\Rightarrow \{f > a\} \cap N \in \mathcal{M}$$

Since X is complete \Rightarrow

$$\{g > a\} \cap (X \setminus N) \in \mathcal{M}$$

$$\Rightarrow \{g > a\} \in \mathcal{M}$$

If we define:

$$f \sim g \quad \text{if} \quad f = g \quad \mu\text{-almost everywhere}$$

Then \sim defines an equivalence relation and hence a function can be regarded as an equivalence class of functions.

(If (X, \mathcal{M}, μ) is complete).

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Limits of Measurable functions.

In this section all functions are $\overline{\mathbb{R}}$ -valued.

Def: (X, \mathcal{M}, μ) measure space
 $\{f_i\}$ measurable on X .

The upper and lower envelopes of $\{f_i\}$ are defined:

$$\sup_i f_i(x) = \sup \{f_i(x) : i = 1, 2, \dots\}$$

$$\inf_i f_i(x) = \inf \{f_i(x) : i = 1, 2, \dots\}$$

Def.

$$\left(\limsup_{i \rightarrow \infty} f_i \right)(x) = \inf_{j \geq 1} \left(\sup_{i \geq j} f_i(x) \right)$$

$$\left(\liminf_{i \rightarrow \infty} f_i \right)(x) = \sup_{j \geq 1} \left(\inf_{i \geq j} f_i(x) \right)$$

Thm: $\sup_i f_i$, $\inf_i f_i$,

$\limsup_{i \rightarrow \infty} f_i$, $\liminf_{i \rightarrow \infty} f_i$

are all measurable.

Proof: Let $a \in \mathbb{R}$.

$$\{x : (\sup_i f_i) > a\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > a\}$$

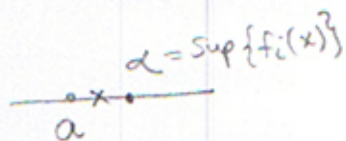
A

B

Let $x \in A$

$$\Leftrightarrow (\sup_i f_i)(x) > a$$

$$\Leftrightarrow \sup \{f_i(x)\} > a$$



$$\Leftrightarrow f_j(x) > a, \text{ for some } j$$

$$\Leftrightarrow x \in \{x : f_j(x) > a\}, \text{ some } j$$

$$\Leftrightarrow x \in B$$

$$\inf_i f_i(x) = -\sup_i (-f_i(x))$$

Def: Let $\{f_i\}$ be a sequence of measurable functions, such that.

$$\lim_{i \rightarrow \infty} f_i(x) = f(x)$$

for μ -almost every $x \in X$.

Then we say:

" f_i converges pointwise almost everywhere"

or

" f_i converges pointwise a.e." ; to f .

Thm: (Egoroff).

Let (X, \mathcal{M}, μ) finite measure space

Let $\{f_j\}$ and $\{f\}$ be measurable and finite almost everywhere on X .

Assume $f_j \rightarrow f$ pointwise a.e

Then:

$\forall \epsilon > 0$, $\exists A \in \mathcal{M}$, s.t. $\mu(A^c) < \epsilon$ and

$f_j \rightarrow f$ uniformly on A .