<u>Proof</u>:

Fix $\varepsilon, \delta > 0$

Let

$$A_0 = \{ x \in X : -\infty < f_j(x) < \infty, \\ -\infty < f(x) < \infty, \text{ and } \\ f_j(x) \longrightarrow f(x) \}.$$

Then $\mu(A_0^c) = 0$

Define:

$$A_i = A_0 \cap \{ x : |f_j(x) - f(x)| < \delta, \ \forall j \geq i \}$$

Note $A_i \subset A_{i+1}$

$x \in A_i \Rightarrow$

$|f_j(x) - f(x)| < \delta, \ \forall j \geq i$

$\therefore |f_j(x) - f(x)| < \delta, \ \forall j \geq i+1$

$\Rightarrow x \in A_{i+1}$

$\therefore A_{i+1}^c \subset A_i^c$

Note:

$$A_0 = \bigcup_{i=1}^{\infty} A_i$$

Because:

Clearly, $A_i \subset A_0 \quad \forall i$

Let $x \in A_0$

$\therefore \quad f_j(x) \to f(x)$

$\therefore \quad \exists N$ s.t

$$|f_j(x) - f(x)| < \delta \quad \forall j \geq N$$

$\Rightarrow \quad x_0 \in A_N$

$\Rightarrow \quad x \in \bigcup_{i=1}^{\infty} A_i$

$$0 = \mu(A_0^c) = \mu\left(\bigcap_{i=1}^{\infty} A_i^c\right)$$

$$= \lim_{i \to \infty} \mu(A_i^c)$$

$\therefore \quad \exists N(\delta)$ s.t. $\mu(A_{N(\delta)}^c) < \varepsilon$

"Given ε, δ, $\exists$ a set $A_{N(\delta)}$ s.t

$\mu(A_{N(\delta)}^c) < \varepsilon$ and $|f_j(x) - f(x)| < \delta \quad \forall j \geq N(\delta)$, $\forall x \in A_{N(\delta)}$"

Apply previous argument to:

$$\frac{\varepsilon}{2^K} \quad \text{and} \quad \delta = \frac{1}{K}, \quad K = 1, 2, \dots$$

Thus, for each $\delta = \frac{1}{K}$ and $\frac{\varepsilon}{2^K}$,

there exists a measurable set $A_{N(K)}$ s.t.

$$\boxed{\mu\left(A_{N(K)}^c\right) < \frac{\varepsilon}{2^K} \text{ and } |f_j(x) - f(x)| < \frac{1}{K} \quad \forall x \in A_{N(K)} \quad \forall j \geq N(K)}$$

<u>Def</u>:

$$A = \bigcap_{K=1}^{\infty} A_{N(K)}$$

$$\mu(A^c) = \mu\left(\bigcup_{K=1}^{\infty} A_{N(K)}^c\right) \leq \sum_{K=1}^{\infty} \frac{\varepsilon}{2^K} = \varepsilon.$$

Let $\varepsilon > 0$, $\Rightarrow \exists K_0$ s.t $\frac{1}{K_0} < \varepsilon$. If $x \in A$, then $x \in A_{N(K_0)}$

$$\therefore \quad |f_j(x) - f(x)| < \frac{1}{K_0} < \varepsilon \quad \forall j \geq N(K_0).$$

Corollary: In the previous theorem, if it is assumed in addition that X is a metric space and μ is a Borel measure with $\mu(X)<\infty$. Then A can be taken as a closed set.

<u>Proof</u>: By Egoroff's theorem

$\exists A \in \mathcal{M}$ s.t.

$f_j \to f$ uniformly on A

and $\mu(A^c) < \frac{\varepsilon}{2}$

<u>Thm</u>: 120.1 $(X, \mathcal{M}, \mu)$ a measure space

X metric space

$\mathcal{M}$ Borel σ-algebra

$\mu(X) < \infty$. Let $\varepsilon > 0$

Then, for every $B \in \mathcal{M}$:

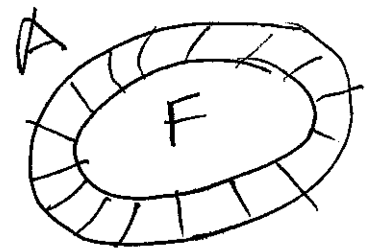
$\exists U$ open, $B \subset U$ s.t. $\mu(U \setminus B) < \varepsilon$

$\exists F$ closed, $F \subset B$ s.t $\mu(B \setminus F) < \varepsilon$.

From Thm 120.1:

$\exists F$ closed $F \subset A$ s.t.

$$\mu(A \setminus F) < \frac{\varepsilon}{2}$$



$$F^c = A^c \cup (A \setminus F)$$

$$\mu(F^c) = \mu(A^c) + \mu(A \setminus F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$

$\boxed{f_j \to f \text{ uniformly on } F \text{ and } \mu(F^c) < \varepsilon}$

Def: $f_i \to f$ almost uniformly if $\forall \varepsilon > 0$, $\exists A \in \mathcal{M}$ s.t.

$\mu(A^c) < \varepsilon$ and $f_i \to f$ unif. on A.

Ex: $f_i = \chi_{[i,\infty)}$ $(\mathbb{R}, \mathcal{M}, \lambda)$

$f_i \to 0$ pointwise

$f_i \not\to 0$ unif. on any set A

s.t. $\mu(A^c) < \infty$.

Def: Let $(X, \mathcal{M}, \mu)$

$\{f_i\}$ measurable functions on X.
f meas.

$f_i \to f$ in measure

if $\forall \varepsilon > 0$:

$$\lim_{i \to \infty} \mu(X \cap \{x : |f_i(x) - f(x)| \geq \varepsilon\}) = 0$$

Thm $(X, \mathcal{M}, \mu)$ finite measure space

$\{f_i\}$, f measurable, finite a.e. on X

If $f_i \to f$ pointwise a.e. on X

$\Rightarrow$ $f_i \to f$ in measure.

Proof:

Fix $\varepsilon, \tilde{\varepsilon} > 0$.

For the pair $\varepsilon, \tilde{\varepsilon}$, $\exists A \in \mathcal{M}, \exists N$ s.t:

$\mu(A^c) < \varepsilon$, $\quad |f_i(x) - f(x)| < \tilde{\varepsilon} \quad \forall x \in A$
$\forall i \geq N$

$\therefore$

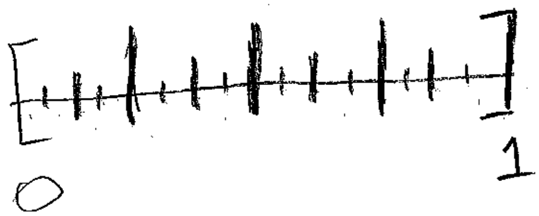
$$\mu(\{x : |f_i(x) - f(x)| \geq \tilde{\varepsilon}\}) \leq \mu(A^c) < \varepsilon$$

$\therefore$ $\mu(\{|f_i(x) - f(x)| \geq \tilde{\varepsilon}\}) < \varepsilon \quad \forall i \geq N$

$\therefore$ $\mu(\{|f_i(x) - f(x)| \geq \tilde{\varepsilon}\}) \longrightarrow 0$.

The converse is not true.

Let $X = [0,1]$, $\mu = \lambda$.



Order the intervals

$I_1, I_2, I_3, \ldots I_k \ldots$

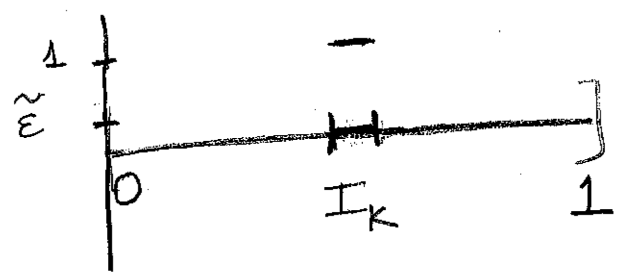
Define

$$f_k = \chi_{I_k}$$

$f_k \to 0$ in measure:

Fix $\varepsilon > 0$, $\tilde{\varepsilon} > 0$

$$\lambda\left(\{x : |f_k(x)| \geq \tilde{\varepsilon}\}\right) < \varepsilon, \quad \forall k \geq N(\varepsilon)$$



Note that $f_k(x) \not\to 0$, for any $x \in [0,1]$.

<u>Thm</u> Let $(X, \mathcal{M}, \mu)$ be a measure space.

$\{f_i\}, f$ measurable functions

$f_i \to f$ in measure.

Then:

$\exists$ subsequence $\{f_{i_k}\}$ s.t.

$$\lim_{k\to\infty} f_{i_k}(x) = f(x).$$

for μ-a.e. $x \in X$.

<u>Proof:</u> Let i_1 be such that:

$$\mu\left(\{x : |f_{i_1}(x) - f(x)| \geq 1\}\right) < \frac{1}{2}$$

We can choose now $i_2 > i_1$ s.t.

$$\mu\left(\{x : |f_{i_2}(x) - f(x)| \geq \frac{1}{2}\}\right) \leq \frac{1}{2^2}$$

Choose now $i_3 > i_2$ s.t

$$\mu\left(\{x : |f_{i_3}(x) - f(x)| \geq \frac{1}{3}\right) \leq \frac{1}{2^3}$$

$\vdots$

$\vdots$

Choose $i_K > i_{K-1}$ s.t

$$\mu\left(\{x : |f_{i_K}(x) - f(x)| \geq \frac{1}{K}\right) \leq \frac{1}{2^K}$$

$\vdots$

Def.

$$A_j = \bigcup_{K=j}^{\infty} \{x : |f_{i_K}(x) - f(x)| \geq \frac{1}{K}\}$$

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

$$\mu(A_1) \leq \mu\left(\bigcup_{K=1}^{\infty} \{x : |f_{i_K}(x) - f(x)| \geq \frac{1}{K}\}\right)$$

$$\leq 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$< \infty$$

Def.

$$B = \bigcap_{j=1}^{\infty} A_j$$

$$\mu(B) = \lim_{j \to \infty} \mu(A_j) \leq \lim_{j \to \infty} \sum_{K=j}^{\infty} \frac{1}{2^K}$$

$$= \lim_{j \to \infty} \frac{1}{2^{j-1}} = 0.$$

Let $x \in B^c = \bigcup_{j=1}^{\infty} A_j^{\cdot c}$

$\Rightarrow \exists \, j(x)$ s.t.

$$x \in \bigcap_{k=j(x)}^{\infty} \{ |f_{i_k}(y) - f(y)| < \tfrac{1}{k} \}$$

Let $\varepsilon > 0$.

Then

$$|f_{i_k}(x) - f(x)| \leq \varepsilon \quad \forall k \geq k_0$$

with

$\frac{1}{k_0} \leq \varepsilon$

$\therefore$ $f_{i_k} \to f$ pointwise almost everywhere.