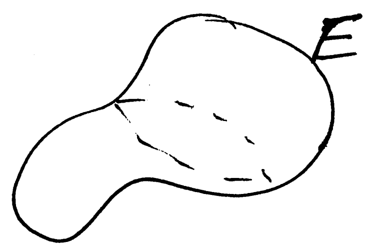
Lesson 2

Recall that our goal is to define the volume of sets in $\mathbb{R}^3$.



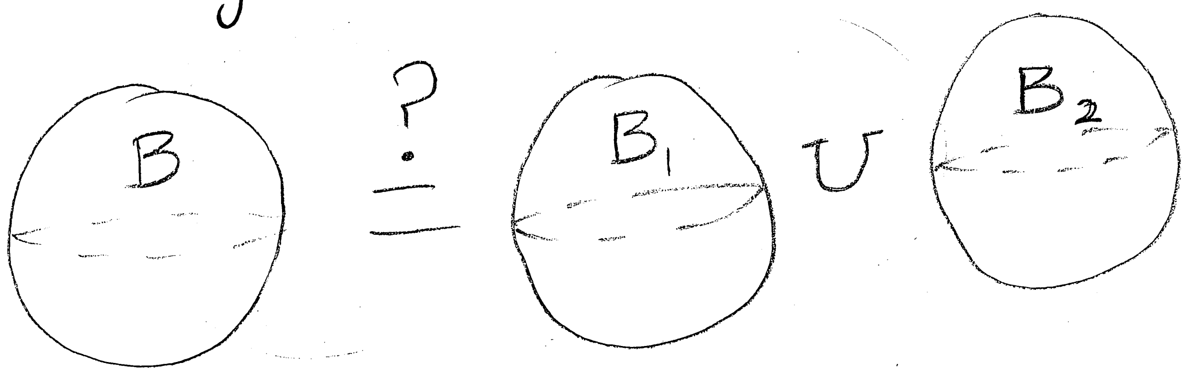
$V(E)$

We would like to define a function $V: \mathcal{P}(\mathbb{R}^3) \longrightarrow [0,\infty]$ such that $V(E)$ gives the volume of E. But this function should satisfy an additivity property. That is, if $\{E_i\}$ is a collection of disjoint sets in $\mathbb{R}^3$, we should have that the volume of $\bigcup_{i=1}^{\infty} E_i$ is the sum of the volumes $V(E_i)$; i.e:

$$V\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} V(E_i). \rightarrow (2)$$

With this requirement, we won't be

able to find such function V. Indeed, it was proved by Banach–Tarski that it is possible to decompose a ball in $\mathbb{R}^3$ in to 6 pieces which can be reassembled by rigid motions to form two balls, each the same size as the original.



$$B = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$$

Assume such V exists. Then, using the left side of the above diagram:

$$V\left(\bigcup_{i=1}^{6} E_i\right) = V(B) = \sum_{i=1}^{6} V(E_i)$$

Using now the right side of the above diagram:

$$V(B_1) + V(B_2) = \sum_j V(E_j) + \sum_k V(E_k)$$

$$= \sum_{i=1}^{6} V(E_i) = V(B)$$

$\therefore$

$$V(B) = V(B_1) + V(B_2),$$

which is not true since B_1 and B_2 are of the same size as B and hence V should give:

$$V(B_1) + V(B_2) = 2V(B).$$

From here the following follows:

(a) We should ask only sub-additivity in (2). That is,

$$V\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} V(E_i)$$

and,

(b) After excluding all pathological sets in the decomposition of B, we should have additivity in the remaining sets.

Therefore, from (a) we define the following:

$\underline{\text{Def}}$: $\varphi : \mathcal{P}(X) \to [0,\infty]$ is an outer measure if

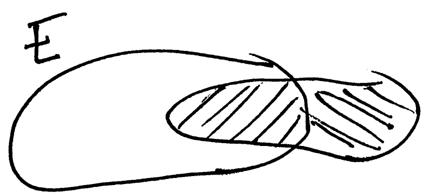
* $\varphi(\emptyset) = 0$
* $\varphi(A_1) \le \varphi(A_2)$, $A_1 \subset A_2$
* $\varphi\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \varphi(A_i)$

Now, from (b), we will prove the additivity property on the sets that are φ- measurable.

Let φ be an outer measure.
We recall the definition of φ-measurable sets.

<u>Def</u>: A set $E \subset X$ is called φ-measurable if

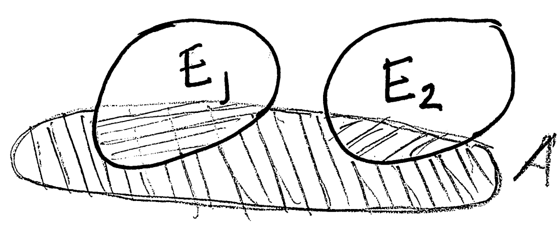
$$\varphi(A) = \varphi(A \cap E) + \varphi(A \cap E^c), \quad \forall A \subset X.$$



This definition will allow us to prove the additivity property for φ. Indeed, if E_1, E_2 are φ-measurable, $E_1 \cap E_2 = \phi$ we have:

$$\varphi(A) = \varphi(A \cap E_2) + \varphi(A \cap E_2^c)$$
$$= \varphi(A \cap E_2) + \varphi(A \cap E_2^c \cap E_1) + \varphi(A \cap E_2^c \cap E_1^c)$$

(1) $$= \varphi(A \cap E_2) + \varphi(A \cap E_1) + \varphi(A \cap (E_1 \cup E_2)^c)$$



With $A := E_1 \cup E_2$ 2.6

$$\boxed{\begin{aligned}\varphi(E_1 \cup E_2) &= \varphi(E_1) + \varphi(E_2) + \varphi(\emptyset) \\ &= \varphi(E_1) + \varphi(E_2)\end{aligned}}$$

From (1), by monotonicity

$$\begin{aligned}\varphi(A) &= \varphi(A \cap E_1) + \varphi(A \cap E_2) + \varphi(A \cap (E_1 \cup E_2)^c) \\ &\geq \varphi((A \cap E_1) \cup (A \cap E_2)) + \varphi(A \cap (E_1 \cup E_2)^c) \\ &= \varphi(A \cap (E_1 \cup E_2)) + \varphi(A \cap (E_1 \cup E_2)^c)\end{aligned}$$

$$\therefore \varphi(A) \geq \varphi(A \cap (E_1 \cup E_2)) + \varphi(A \cap (E_1 \cup E_2)^c),$$

and hence $E_1 \cup E_2$ is φ-measurable.

<u>Remark</u>: Since $A = (A \cap E) \cup (A \cap E^c)$, and by monotonicity it is always true that:

$$\varphi(A) = \varphi[(A \cap E) \cup (A \cap E^c)] \leq \varphi(A \cap E) + \varphi(A \cap E^c)$$

then, in order to check if E is φ-measurable, we only need to prove that:

$$\varphi(A) \geq \varphi(A \cap E) + \varphi(A \setminus E), \quad \forall A \subset X.$$

Another way to check if E is measurable is to use the following:

<u>Lemma</u>: $E \subset X$ is φ-measurable if and only if

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q)$$

for any $P \subset E$, $Q \subset E^c$

<u>Proof</u>: $\Leftarrow$ Let $A \subset X$

Define $P := A \cap E$, $Q = A \setminus E$

$\therefore$ $P \subset E$, $Q \subset E^c$

Then

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q)$$

Since $A = P \cup Q$, then

$$\varphi(A) = \varphi(A \cap E) + \varphi(A \setminus E)$$

$\therefore$ E is measurable.

2.8

$\Rightarrow$ Assume E is measurable.

Let P, Q, $P \subset E$, $Q \subset E^c$

Then,

$$\varphi(P \cup Q) = \varphi((P \cup Q) \cap E) + \varphi((P \cup Q) \setminus E)$$

$$= \varphi((P \cap E) \cup (Q \cap E)) + \varphi((P \cap E^c) \cup (Q \cap E^c))$$

$$= \varphi(P \cup \emptyset) + \varphi(\emptyset \cup Q)$$

$$= \varphi(P) + \varphi(Q).$$

∎

2.2 Cardinal numbers.

<u>Def</u>, A and B are said to be equivalent if $\exists f: A \to B$ 1-1 and on-to. We write

$$A \sim B$$

$\sim$ defines an equivalent relation on X.

$\sim$ is reflexive, symmetric & transitive

Two sets in the same equivalence class are said to have the same cardinal number or to be of the same cardinality.

<u>Def</u>: $\text{Card}\{1, 2, \cdots, n\} = n$

$\text{Card } \mathbb{N} = \aleph_0$

$\text{Card } \mathbb{R} = c$

Def:

A is finite if Card $A = n$

If A is not finite the it is called infinite set

If $A \sim \mathbb{N}$ then A is said to be denumerable.

If Card $A = n$ or Card $A = \mathbb{N}$ then A is called countable. Otherwise it is called uncountable.

In other books: A is countable if $A \sim \mathbb{N}$. And A is at most countable if A is finite or countable.

Def: Let $\alpha = \text{Card } A$, $\beta = \text{Card } B$.

We say $\alpha \leq \beta$ if and only if $\exists\, B_1 \subset B$ s.t. $A \sim B_1$.

We say $\alpha < \beta$ if there exists no set $A_1 \subset A$ s.t. $A_1 \sim B$.

<u>Thm</u>: $\alpha \le \beta$ and $\beta \le \alpha \Rightarrow \alpha = \beta$

<u>Def</u>: $\alpha + \beta = \text{Card}(A \cup B)$, $A \cap B = \emptyset$

$\alpha^\beta = \text{Card } F$

where F is the family of all functions $f: B \to A$.

We also have

$$\aleph_0 + \aleph_0 = \aleph_0$$

$$c + c = c$$

$$2^{\aleph_0} = c$$

$$2^\alpha > \alpha$$