6.4 L^p spaces.

<u>Def.</u> Let $1 \le p \le \infty$.
$E \in \mathcal{M}$.

$L^p(E, \mathcal{M}, \mu)$ denote the class of all measurable functions f on E such that

$$\|f\|_{p,E;\mu} < \infty.$$

$$\|f\|_{p,E;\mu} = \begin{cases} \left(\int_E |f|^p \, d\mu\right)^{1/p}, & 1 \le p < \infty \\ \inf\{M : |f| \le M \ \mu\text{-a.e. on } E\}, & p = \infty. \end{cases}$$

Note:

(i) $\|f\|_p \ge 0$ for any measurable f.

(ii) $\|f\|_p = 0 \iff f = 0$ μ-a.e.

$$\left(\int_E |f|^p \, d\mu\right)^{1/p} = 0 \iff \int_E |f|^p = 0$$

$$\iff |f|^p = 0 \ \mu\text{-a.e.}$$

$$\iff f = 0 \ \mu\text{-a.e.}$$

For $p=\infty$,

Suppose $|f|>0$ on a set of positive measure $P \subset E$, $\mu(P)>0$.

Let
$$P_n = \{x \in P : |f(x)| \geq \frac{1}{n}\}$$
$$\Rightarrow P = \bigcup_{n=1}^{\infty} P_n.$$

Since $\mu(P)>0$, then $\exists N$ s.t.
$$\mu(A_N)>0$$
$$\therefore |f| \geq \frac{1}{N} \quad \text{on } A_N \qquad (1)$$

By definition of infimum, $\exists M_k$ such that
$$M_k \longrightarrow \|f\|_\infty = 0$$
and
$$|f(x)| \leq M_k \quad \mu\text{-a.e. } x$$
Thus, $\exists K_0$ such that
$$0 < M_{K_0} < \frac{1}{N}$$
Thus
$$|f(x)| \leq M_{K_0} < \frac{1}{N} \text{ for } \mu\text{-a.e. } x,$$
which contradicts (1)

(iii) $\|cf\|_p = |c|\,\|f\|_p$ for any $c \in \mathbb{R}$.

Notation: $L^p(X)$ instead of $L^p(X, \mathcal{M}, \mu)$

If X is a topological space:

$L^p_{loc}(X) = \{ f : f \in L^p(K),$ for each compact set $K \subset X \}$.

<u>Lemma</u>: $L^p(X)$ is a vector space.

Let $1 \le p \le \infty$.

(i) $f, g \in L^p(X) \Rightarrow f+g \in L^p(X)$

(ii) $f \in L^p(X)$, $c \in \mathbb{R} \Rightarrow cf \in L^p(X)$

<u>Proof</u>: Let $1 \le p < \infty$. Then, for any $a, b \in \mathbb{R}$, $a, b > 0$

$$\left(\frac{a+b}{2} \right)^p \le \frac{1}{2}(a^p + b^p)$$

This is because:

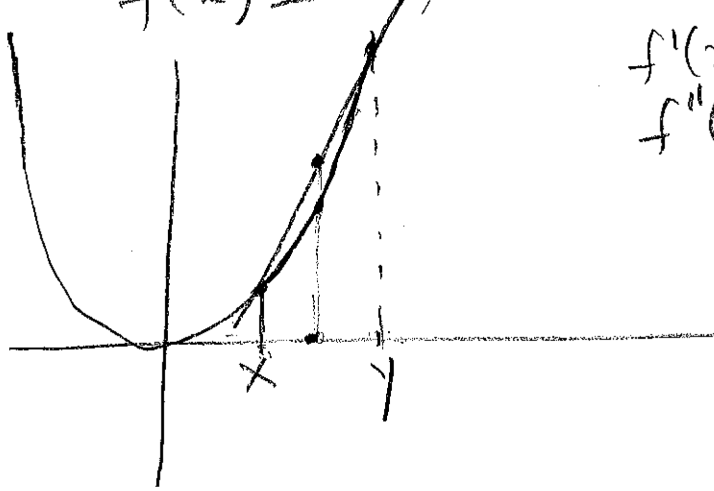
$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = x^p \text{ is convex}$$

<u>Note:</u> $f : (a, b) \to \mathbb{R}$ is convex if

$$f[(1-t)x + ty] \le (1-t)f(x) + t f(y)$$

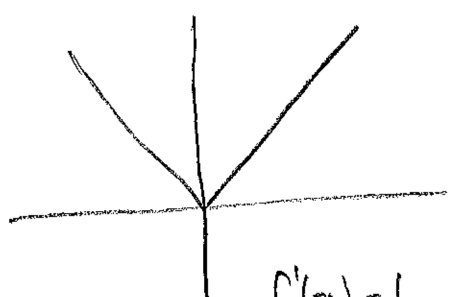
$$\forall (x, y) \in (a, b), \quad t \in [0, 1]$$

<u>Ex:</u> $f(x) = x^2$,



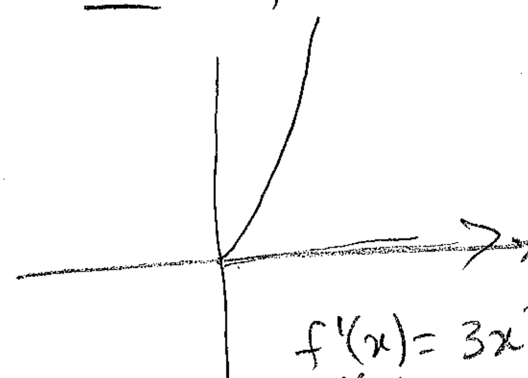
$f'(x) = 2x$
$f''(x) = 2 \ge 0$

<u>Ex</u> $f(x) = x$



$f'(x) = 1$
$f''(x) = 0$

<u>Ex:</u> $f(x) = x^3$



$f'(x) = 3x^2$
$f''(x) = 6x$.

From the definition of convex function, if $t = \frac{1}{2}$:

$$\left(\frac{a+b}{2}\right)^p \le \frac{1}{2}(a^p + b^p)$$

$\Rightarrow$ For μ-a.e. X:

$$|f(x)+g(x)|^p$$

$$\le (|f(x)| + |g(x)|)^p$$

$$\le \frac{2^p}{2}(|f(x)|^p + |g(x)|^p)$$

$$= 2^{p-1}(|f(x)|^p + |g(x)|^p), \quad \mu\text{-a.e}$$

$$\Rightarrow \int_X |f+g|^p \le 2^{p-1}\left[\int_X |f|^p + \int_X |g|^p\right]$$

$$< \infty \quad \Rightarrow \quad f+g \in L^p(X)$$

For $p = \infty$, we have $\exists M, N$ s.t:

$$|f(x)| \le M \quad \mu\text{-a.e.}, \quad |g(x)| \le N \quad \mu\text{-a.e.}$$

$$\Rightarrow |f(x)+g(x)| \le |f(x)| + |g(x)| \le M+N \quad \mu\text{-a.e}$$

$$\Rightarrow f+g \in L^\infty(X)$$

Lemma:

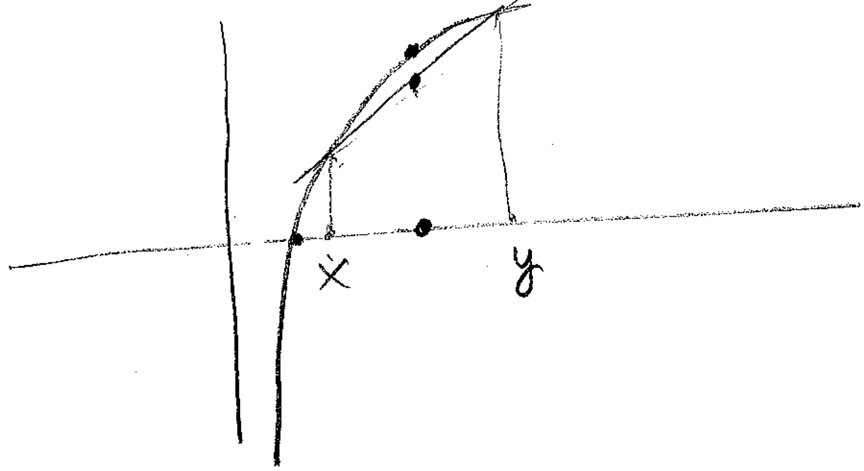
$$1 < p < \infty$$

$$\frac{1}{p} + \frac{1}{p'} = 1$$

$$a, b \geq 0$$

$$\Rightarrow \quad ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

Proof: $f(x) = \ln x$ is strictly concave



$$f''(x) = -\frac{1}{x^2} < 0$$

$x, y > 0$, $\ln(\lambda x + (1-\lambda) y) > \lambda \ln x + (1-\lambda) \ln y$

let $x = a^p$, $y = b^{p'}$, $\lambda = \frac{1}{p}$, $1 - \lambda = \frac{1}{p'}$

$$\Rightarrow \ln\left(\frac{1}{p} a^p + \frac{1}{p'} b^{p'}\right) > \frac{1}{p} \ln(a^p) + \frac{1}{p'} \ln(b^{p'})$$

$$= \ln (a^p)^{1/p} + \ln (b^{p'})^{1/p'}$$

$$= \ln(ab)$$

$\Rightarrow$

$$\ln(ab) < \ln\left(\frac{1}{p}a^p + \frac{1}{p'}b^{p'}\right)$$

$$\Rightarrow \quad ab < \frac{1}{p}a^p + \frac{1}{p'}b^{p'}$$

Equality holds $\Longleftrightarrow a^p = b^{p'}$

Thm. (Hölder's inequality).

$1 \le p \le \infty$.

f, g measurable functions

$\Rightarrow$

$$\int_X |fg|\,d\mu = \int_X |f|\,|g|\,d\mu \le \|f\|_p \|g\|_{p'}$$

Equality holds (for $1<p<\infty$), if and only if.

$$\|f\|_p^p \, |g|^{p'} = \|g\|_{p'}^{p'} \, |f|^p \quad \mu\text{-a.e.}$$

Proof: $p = 1$, $p' = \infty$, $\|f\|_1 < \infty$, $\|g\|_\infty < \infty$

$|f(x)|\,|g(x)| \le M_n |f(x)|$, $\forall x \in A_n$

$\mu(A_n^c) = 0$, $M_n \longrightarrow \|g\|_\infty$

$$|f(x)|\,|g(x)| \leq M_n\,|f(x)| \quad , \; n = 1, 2, \ldots$$
$$\forall x \in \bigcap_{n=1}^{\infty} A_n$$

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \mu\left(\bigcup_{n=1}^{\infty} A_n^c\right)$$
$$\leq \sum_{n=1}^{\infty} \mu(A_n^c) = 0$$

$$\therefore \quad |f|\,|g| \leq \|g\|_\infty |f| \quad \mu\text{-a.e.}$$

$$\Rightarrow \int_X |f|\,|g| \leq \int_X \|g\|_\infty |f|$$

$$\Rightarrow \int_X |fg|\,d\mu = \int_X |f|\,|g|\,d\mu$$
$$\leq \int_X \|g\|_\infty |f|\,d\mu$$
$$= \|g\|_\infty \int_X |f|\,d\mu = \|g\|_\infty \|f\|_{L^1}$$

$$\Rightarrow \int_X |fg| \leq \|f\|_{L^1} \|g\|_\infty$$

$1 < p < \infty$.

Assume $0 < \|f\|_p, \|g\|_{p'} < \infty$

If $\|f\|_p = 0$ or ∞ it is clear

If $\|g\|_{p'} = 0$ or ∞, result is clear.

$$\Rightarrow \int_X \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_{p'}} \, d\mu$$

$$\leq \int_X \left(\frac{|f|^p}{p\|f\|_p^p} + \frac{|g|^{p'}}{p'\|g\|_{p'}^{p'}} \right) d\mu.$$

$$= \frac{1}{p\|f\|_p^p} \int_X |f|^p d\mu + \frac{1}{p'\|g\|_{p'}^{p'}} \int_X |g|^{p'} d\mu$$

$$= \frac{\cancel{\|f\|_p^p}}{p\cancel{\|f\|_p^p}} + \frac{\cancel{\|g\|_{p'}^{p'}}}{p'\cancel{\|g\|_{p'}^{p'}}} = \frac{1}{p} + \frac{1}{p'} = 1$$

$$\Rightarrow \int_X \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_{p'}} \, d\mu \leq 1$$

$$\Rightarrow \int_X |fg| \, d\mu = \int_X |f| \, |g| \, d\mu$$

$$\leq \|f\|_p \|g\|_{p'}$$