197

Thm: Let $(X, \mathcal{M}, \mu)$ σ- finite measure space

f measurable, $1 \leq p \leq \infty$,

$\frac{1}{p} + \frac{1}{p'} = 1$,

Then,

$$\|f\|_p = \sup\left\{\int_X fg\,d\mu : \|g\|_{p'} \leq 1\right\}.$$

Proof:

$$\int fg\,d\mu \leq \|f\|_p \|g\|_{p'} \leq \|f\|_p.$$

$\forall g \in L^{p'}$, $\|g\|_{p'} \leq 1$

$$\Rightarrow \boxed{\sup\left\{\int fg\,d\mu : \|g\|_{p'} \leq 1\right\} \leq \|f\|_p}$$

Need opposite inequality:

$$\boxed{\|f\|_p \leq \sup\left\{\int fg\,d\mu : \|g\|_{p'} \leq 1\right\}}$$

<u>Case $p=1$:</u>

Let $g = \text{sign}(f)$, $g(x) = \begin{cases} 1, & \text{if } f(x) \geq 0 \\ -1, & \text{if } f(x) < 0 \end{cases}$

$\Rightarrow \|g\|_\infty \leq 1$

$$\int_X fg\,d\mu = \int_X |f|\,d\mu = \|f\|_1$$

$$\Rightarrow \boxed{\|f\|_1 \leq \sup\left\{ \int fg\,d\mu : \|g\|_{p'} \leq 1 \right\}}$$

<u>Case $1 < p < \infty$:</u>

If $\|f\|_p = 0$ it is clear. Assume that

$0 < \|f\|_p < \infty$,

Let $$g = \frac{|f|^{p/p'}\,\text{Sign}(f)}{\|f\|_p^{p/p'}}$$

$\frac{1}{p} + \frac{1}{p'} = 1$

$p' + p = pp'$

$1 + \frac{p}{p'} = p$

Then $\|g\|_{p'} = 1$ and:

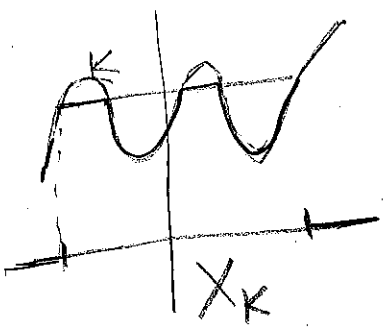
$$\int_X fg\,d\mu = \frac{1}{\|f\|_p^{p/p'}} \int_X |f|^{p/p'+1}\,d\mu = \frac{\|f\|_p^p}{\|f\|_p^{p/p'}} = \|f\|_p$$

$$\Rightarrow \boxed{\|f\|_p \leq \operatorname{Sup}\left\{\int_X fg, \quad \|g\|_{p'} \leq 1\right\}}$$

Assume now $\|f\|_p = \infty$

$$X = \bigcup_{K=1}^{\infty} X_K, \quad \mu(X_K) < \infty$$

Def



$$h_K(x) = \chi_{X_K} \min(|f(x)|, K), \quad x \in X$$

$$\Rightarrow h_K \in L^p(X)$$

$$\left(\int_X h_K^p\right)^{1/p} \leq \left(\int_X K^p \chi_{X_K} d\mu\right)^{1/p}$$

$$= K \int_{X_K} d\mu = K\mu(X_K)$$

$$\Rightarrow \|h_K\|_p < \infty$$

$$\|h_K\|_p \longrightarrow \infty$$

[Because Mon. Conv. Thm $\Rightarrow$ $\int_X h_K^p \, d\mu \rightarrow \int_X |f|^p d\mu$]

WLOG $\|h_k\|_p > 0$

$\exists\, g_k \in L^{p'}(X)$, $\|g_k\|_{p'} = 1$ s.t.

$$\int_X h_k g_k \, d\mu = \|h_k\|_p \longrightarrow \infty$$

$$h_k \geq 0 \implies g_k \geq 0$$

$$\int_X f \underbrace{(\operatorname{sign} f)\, g_k}_{g} \, d\mu = \int_X |f|\, g_k \, d\mu$$

$$\geq \int_X h_k g_k \, d\mu \longrightarrow \infty$$

$$\therefore \quad \sup\left\{ \int_X f g \, d\mu : \|g\|_{p'} \leq 1 \right\} = \infty = \|f\|_p.$$

Case $p = \infty$:

If $\|f\|_\infty = 0$, it is clear.

Suppose $M := \sup\{\int fg \, d\mu : \|g\|_1 \leq 1\} < \|f\|_\infty$

Thus, $\exists\, \varepsilon > 0$ such that $0 < M + \varepsilon < \|f\|_\infty$.

Define: $E_\varepsilon := \{x : |f(x)| \geq M + \varepsilon\}$

Note that E_ε has positive measure, for otherwise we would have

$$\|f\|_\infty \leq M+\varepsilon .$$

Since μ is σ-finite then $X = \bigcup_{k=1}^{\infty} X_k$, $\mu(X_k) < \infty$. Then $\exists\, K_0$ s.t. $0 < \mu(E_\varepsilon \cap X_{K_0}) < \infty$.

Let $E := X_{K_0}$. Thus $0 < \mu(E_\varepsilon \cap E) < \infty$.

Define :

$$g_\varepsilon = \frac{1}{\mu(E_\varepsilon \cap E)} \chi_{E_\varepsilon \cap E} \operatorname{sign}(f)$$

Thus, $\|g_\varepsilon\|_1 = 1$ and

$$\int f g_\varepsilon \, d\mu = \frac{1}{\mu(E_\varepsilon \cap E)} \int_{E_\varepsilon \cap E} |f| \, d\mu \geq M+\varepsilon$$

Thus

$$M+\varepsilon \leq \int f g_\varepsilon \leq \sup\left\{ \int f g \, d\mu : \|g\|_1 \leq 1 \right\} = M = \|f\|_\infty ;$$

which is a contradiction.

We conclude that:

$$\sup\left\{ \int f g \, d\mu : \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

Thm (Minkowski's inequality):

$$1 \le p \le \infty$$
$$f, g \in L^p(X)$$
$$\Rightarrow \quad \|f+g\|_p \le \|f\|_p + \|g\|_p$$

Proof. $p=1$ is clear, $p=\infty$ is clear $\Big\}$ triangle inequality.

$\|f+g\|_p = 0$ is clear.

Suppose $1 < p < \infty$, $\|f+g\|_p > 0$.

$$\|f+g\|_p^p = \int |f+g|^p d\mu$$
$$= \int |f+g|^{p-1} |f+g| \, d\mu$$
$$\le \int |f+g|^{p-1} |f| \, d\mu + \int |f+g|^{p-1} |g| \, d\mu$$
$$\le \left(\int \left(|f+g|^{p-1}\right)^{p'}\right)^{1/p'} \left(\int |f|^p d\mu\right)^{1/p}$$
$$+ \left(\int \left(|f+g|^{p-1}\right)^{p'}\right)^{1/p'} \left(\int |g|^p d\mu\right)^{1/p}$$

$\frac{1}{p} + \frac{1}{p'} = 1$

$p' + p = pp'$

$\frac{1}{p'} = 1 - \frac{1}{p}$

$\frac{1}{p'} = \frac{p-1}{p}$

$$= \left(\int |f+g|^p\right)^{\frac{p-1}{p}} \left(\int |f|^p d\mu\right)^{1/p}$$
$$+ \left(\int |f+g|^p\right)^{\frac{p-1}{p}} \left(\int |g|^p d\mu\right)^{1/p}$$

$$= \|f+g\|_p^{p-1} \|f\|_p + \|f+g\|_p^{p-1} \|g\|_p$$

$$= \|f+g\|_p^{p-1} \left(\|f\|_p + \|g\|_p \right)$$

$$\Rightarrow \quad \|f+g\|_p \le \|f\|_p + \|g\|_p .$$

204

<u>Thm</u>: $\mu(X) < \infty$

$1 \le p \le q \le \infty$

$L^q(X) \subset L^p(X)$

Case $q < \infty$:

Let $f \in L^q$

$$\Rightarrow \int_X |f|^p \cdot 1 \, d\mu \le \| |f|^p \|_{q/p} \|1\|_{\frac{q}{q-p}}$$

$$\frac{1}{\frac{q}{p}} + \frac{1}{\frac{q}{q-p}} = \frac{p}{q} + \frac{q-p}{q} = \frac{q}{q} = 1$$

$$= \left(\int_X (|f|^p)^{q/p} \right)^{p/q} \cdot \left(\int_X 1 \, d\mu \right)^{\frac{q-p}{q}}$$

$$= \left(\int_X |f|^q \right)^{\frac{p}{q}} \mu(X)^{\frac{q-p}{q}} = \|f\|_q^p \, \mu(X)^{\frac{q-p}{q}}$$

$$< \infty$$

$$\Rightarrow \int_X |f|^p < \infty$$

$$\Rightarrow \left(\int_X |f|^p \right)^{1/p} < \infty$$

Thm: If $1 \le p \le \infty$, then $L^p(X)$ is a complete metric space; i.e, if $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^p(X)$, then there is an $f \in L^p(X)$ s.t.

$$\lim_{k \to \infty} \|f_k - f\|_p = 0$$

Proof. Let $\{f_k\}_{k=1}^{\infty}$ Cauchy in $L^p(X)$, $\forall \varepsilon > 0$

$\Rightarrow \exists N$ s.t

$$\|f_k - f_m\|_p = \rho(f_k, f_m) < 1, \ \forall k, m \ge N.$$

$$\Rightarrow \|f_k\|_p = \|f_k - f_N + f_N\|_p$$

$$\le \|f_k - f_N\|_p + \|f_N\|_p, \quad \forall k \ge N$$

$$< 1 + \|f_N\|, \ \forall k \ge N.$$

$\Rightarrow \{\|f_k\|_p\}_{k=1}^{\infty}$ is bounded

Let $1 \le p < \infty$.

Let $\varepsilon > 0$.

$A_{k,m} = \{x : |f_k(x) - f_m(x)| \geq \varepsilon\}$.

Then:

$$\int_{A_{k,m}} |f_k - f_m|^p \, d\mu \geq \int_{A_{k,m}} \varepsilon^p \, d\mu = \varepsilon^p \mu(A_{k,m})$$

$$\mu\{x : |f_k(x) - f_m(x)| \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \|f_k - f_m\|_{p,X}^p \to 0$$

as $k, m \to \infty$

$\therefore \{f_k\}$ is fundamental in measure

Exercise 5.13 $\Rightarrow \exists f$ s.t

$f_k \to f$ in measure

Thm 146.2 $\Rightarrow \exists \{f_{k_j}\}$ s.t.

$f_{k_j} \to f$ pointwise μ-a.e.

Fatou's Lemma $\Rightarrow$

$$\int |f|^p \, d\mu \le \liminf_{j\to\infty} \int |f_{k_j}|^p \, d\mu < \infty$$

$\Rightarrow \; f \in L^p(X)$

Let $\varepsilon > 0$

$\Rightarrow \exists M$ s.t.

$$\|f_k - f_m\|_p < \varepsilon, \quad \forall k, m \ge M.$$

Fatou's Lemma $\Rightarrow$.

Fix $k \ge M$

$$\int_X |f_k - f|^p \, d\mu$$

$$\le \liminf_{j\to\infty} \int_X |f_k - f_{k_j}|^p \, d\mu$$

$$< \varepsilon^p$$

$$\therefore \int_X |f_k - f|^p \, d\mu < \varepsilon^p \quad \forall k \ge M$$

$\therefore \; f_k \to f$ in $L^p(X)$