Lesson 27

Vitali's Convergence Theorem

We have the following:

__Lemma 1__: Let f be an integrable function. Then, for every $\varepsilon>0$, there exists $\delta>0$ such that

$$A\subset X,\ \mu(A)<\delta \implies \int_A |f|\,d\mu < \varepsilon \qquad (*)$$

__Prove__:

We proceed by contradiction. Thus, $\exists\ \varepsilon>0$ such that for each $\delta=\frac{1}{k}$ $(*)$ is not true. That is, there exists a sequence of sets $\{A_k\}$ such that

$$\mu(A_k)<\frac{1}{2^k} \quad \text{and} \quad \int_A |f|\,d\mu \geq \varepsilon .$$

Define:

$$A=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \limsup A_k \qquad (1)$$

Since $\sum_{k=1}^{\infty}\mu(A)<\infty$ then Borel Cantelli's

Theorem implies that:

$$\mu(A) = 0 \qquad (2),$$

Define:

$$\nu(E) = \int_E |f|\,d\mu, \quad \text{for all } E \in \mathcal{M}.$$

Corollary 156.1 gives that ν is a measure in $(X, \mathcal{M})$.

From (1), since $\nu(A) < \infty$,

$$\nu(A) = \lim_{n\to\infty} \nu\left(\bigcup_{k=n}^{\infty} A_k\right)$$

and since $\nu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \nu(A_n) \geq \varepsilon$, it follows that

$$\nu(A) \geq \varepsilon \qquad (3)$$

But, from (2):

$$\nu(A) = \int_A |f|\,d\mu = 0, \quad \text{because } \mu(A) = 0,$$

which contradicts (3). ∎

<u>Corollary</u>: Let $f \in L^p(X)$, $1 \leq p < \infty$. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.:

$$A \subset X,\ \mu(A) < \delta \implies \int_A |f|^p < \varepsilon$$

Lemma 2: Let f be an integrable function. Then $\forall \varepsilon > 0$ $\exists E \in \mathcal{M}$, $\mu(E) < \infty$ such that

$$\int_{E^c} |f| \, d\mu < \varepsilon$$

Proof: Let

$$A = \{x : |f| \neq 0\}.$$

Note:

$$A = \bigcup_{n=1}^{\infty} A_n, \quad A_n = \{x : |f| \geq \tfrac{1}{n}\}$$

Note:

$$\underline{A_n \subset A_{n+1}}$$

Also, $\underline{\mu(A_n) < \infty}$ for otherwise, if $\mu(A_n) = \infty$ then:

$$\int_X |f| \, d\mu \geq \int_{A_n} |f| \, d\mu \geq \frac{1}{n} \mu(A_n) = \infty,$$

which contradicts that $\int_X |f| \, d\mu < \infty$.

Define

$$f_n = |f| \chi_{A_n}.$$

Then $f_n \uparrow |f| \chi_A$ and the Monotone Convergence

Theorem yields:

(27.4)

$$\int_X |f|\chi_A \, d\mu = \int_A |f| \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \int_{A_n} |f| \, d\mu$$

Note that $\int_A |f| \, d\mu = \int_X |f| \, d\mu$ since $\int_{A^c} |f| \, d\mu = 0$.

Thus, $\exists N$ s.t. $\int_X |f| \, d\mu - \int_{A_N} |f| \, d\mu < \varepsilon$.

Therefore:

$$\int_X |f| \, d\mu = \int_{A_N} |f| \, d\mu + \int_{A_N^c} |f| \, d\mu$$

and

$$\int_{A_N^c} |f| \, d\mu = \int_X |f| \, d\mu - \int_{A_N} |f| \, d\mu < \varepsilon$$

$$\therefore \quad \int_{A_N^c} |f| \, d\mu < \varepsilon. \quad \blacksquare$$

<u>Corollary</u>: Let $f \in L^p(X)$, $1 \le p < \infty$. Then $\forall \varepsilon > 0$, $\exists E \in \mathcal{M}$ such that $\mu(E) < \infty$ and

$$\int_{E^c} |f|^p \, d\mu < \varepsilon$$

Thm: (Vitali's Convergence Thm).

Suppose $\{f_k\}, f \in L^p(X)$, $1 \le p < \infty$.
Then $\|f_k - f\|_p \to 0$ if the following three conditions hold:

(i) $f_k \to f$ μ-a.e.

(ii) $\forall \varepsilon > 0$, $\exists E \in \mathcal{M}$ such that $\mu(E) < \infty$ and
$$\int_{E^c} |f_k|^p \, d\mu < \varepsilon \quad \forall k \in \mathbb{N}.$$

(iii) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
$$\mu(E) < \delta \Rightarrow \int_{E^c} |f_k|^p \, d\mu < \varepsilon \quad \forall k \in \mathbb{N}.$$

Conversely, if $\|f_k - f\|_p \to 0$ then:

(ii) & (iii) hold, and
(i) holds for a subsequence

Proof:

Assume (i), (ii), (iii) hold.

Let $\varepsilon > 0$.

Then $\exists \delta > 0$ given by (iii), and (ii) yields a set E, $\mu(E) < \infty$ such that:

$$\int_{E^c} |f_k|^p \, d\mu < \varepsilon, \quad \forall k.$$

Since $\mu(E) < \infty$, we apply Egoroff's Theorem to obtain a measurable set $B \subset E$ with:

$\mu(E \setminus B) < \delta$ and $f_k \to f$ uniformly on B.

Now write:

$$\int_X |f_k - f|^p \, d\mu = \int_E |f_k - f|^p \, d\mu + \int_{E^c} |f_k - f|^p \, d\mu$$

$$= \int_B |f_k - f|^p \, d\mu + \int_{E \setminus B} |f_k - f|^p \, d\mu + \int_{E^c} |f_k - f|^p \, d\mu$$

Note that:

$$|f_k - f|^p \le 2^{p-1}\left(|f_k|^p + |f|^p\right)$$

$\therefore$

$$\int_X |f_k - f|^p \, d\mu \le \int_B |f_k - f|^p \, d\mu$$

$$+ \, 2^{p-1} \int_{E \setminus B} |f_k|^p \, d\mu + 2^{p-1} \int_{E \setminus B} |f|^p \, d\mu$$

$$+ \, 2^{p-1} \int_{E^c} |f_k|^p + 2^{p-1} \int_{E^c} |f|^p \, d\mu$$

By Fatou's Lemma

$< \varepsilon$ because $f_k \to f$ unif on B

$$\int_X |f_k - f|^p \, d\mu \le \int_B |f_k - f|^p$$

$< \varepsilon$ because of (iii)

$$+ \, 2^{p-1} \int_{E \setminus B} |f_k|^p \, d\mu + 2^{p-1} \liminf_{k \to \infty} \int_{E \setminus B} |f_k|^p \, d\mu$$

$$+ \, 2^{p-1} \int_{E^c} |f_k|^p + 2^{p-1} \underline{\lim} \int_{E^c} |f_k|^p \, d\mu$$

$< \varepsilon$ because of (ii)

$$\le \varepsilon$$

Conversely, suppose now that $\|f_k - f\|_p \to 0$ (27.8)

$\Rightarrow \forall \varepsilon > 0 \ \exists K_0$ s.t.:

$$\|f_k - f\|_p < \frac{\varepsilon}{2} \quad \forall k \geq K_0$$

Lemma 2 $\Rightarrow \exists A, B$, $\mu(A) < \infty$, $\mu(B) < \infty$

s.t:

$$\int_{A^c} |f|^p < \left(\frac{\varepsilon}{2}\right)^p \text{ and } \int_{B^c} |f_k|^p d\mu < \varepsilon^p, \ k = 1, 2, \dots, k_0.$$

Now:

$$\|f_k\|_{p, A^c} \leq \|f_k - f\|_{p, A^c} + \|f\|_{p, A^c}$$
$$< \varepsilon, \quad \forall k > k_0$$

With $E = A \cup B$ we obtain: (ii):

$$\mu(E) < \infty \quad \text{and} \quad \int_{E^c} |f_k|^p < \varepsilon, \quad \forall k \in \mathbb{N}.$$

Lemma 1 $\Rightarrow \exists \delta > 0$ s.t (take $\delta = \min\{\delta_f, \delta_{f_1}, \dots, f_{k_0}\}$):

$$\mu(E) < \delta \Rightarrow \int_E |f|^p < \varepsilon \text{ and } \int_E |f_k|^p d\mu < \varepsilon, \ k = 1, 2, \dots, k_0.$$

and

$$\|f\|_{p,E} \leq \|f_k - f\|_{p,E} + \|f\|_{p,E} < \varepsilon \quad \forall k > k_0$$

$$\therefore \quad \int_E |f_k|^p d\mu < \varepsilon \quad \forall k.$$