6.7 The Dual of L^p.

Def: $(X, \mathcal{M}, \mu)$ measure space.

$F : L^p(X) \to \mathbb{R}$ is a linear functional on $L^p(X)$ if

$$F(af + bg) = aF(f) + bF(g)$$

Def: $F : L^p(X) \to \mathbb{R}$ linear is bounded if

$$\|F\| := \sup\{|F(f)| : f \in L^p(X) : \|f\|_p \leq 1\} < \infty$$

Thm:
$F : L^p(X) \to \mathbb{R}$ linear functional

F bounded $\Longleftrightarrow$ F continuous.

Proof: $\Longrightarrow$

F is bounded; ie. $\|F\| < \infty$.

Let $f \neq 0$, $f \in L^p(X)$, then.

$$\left| F\left(\frac{f}{\|f\|_p} \right) \right| \leq \|F\|$$

$$\therefore \quad |F(f)| \leq \|f\|_p \, \|F\|.$$

$$\Rightarrow |F(f-g)| \leq \|F\| \, \|f-g\|_p \quad \forall f, g \in L^p(X)$$

$\Rightarrow F$ is uniformly continuous on $L^p(X)$

$\Rightarrow F$ is continuous on $L^p(X)$

$\Longleftarrow$ Assume F is continuous on $L^p(X)$

$\therefore$ F is continuos at $f \equiv 0$.

$\therefore$ $\exists \, \delta > 0$ s.t.

$$|F(f)| \leq 1 \quad \forall f \in N_\delta(0)$$

$$\{f : \|f - 0\|_p < \delta\}$$

Let $f \in L^p(X)$, $\|f\|_p > 0$. $\Rightarrow$.

$$|F(f)| = \left| \frac{\|f\|_p}{\delta} F\left(\frac{\delta}{\|f\|_p} f\right) \right|$$

$$= \frac{\|f\|_p}{\delta} \left| F\left(\frac{\delta}{\|f\|_p} f\right) \right|$$

$$\le \frac{\|f\|_p}{\delta} \cdot 1$$

$$\Rightarrow \left| F\left(\frac{f}{\|f\|_p}\right) \right| \le \frac{1}{\delta}$$

$$\Rightarrow \|F\| \le \frac{1}{\delta}$$

$\Rightarrow$ F is bounded.

<u>Definition</u>: Let $\mathcal{B}(L^p(X), \mathbb{R})$ denote the set of all bounded linear mappings of $L^p(X)$ in to $\mathbb{R}$.

<u>Thm</u>: $\mathcal{B}(L^p(X, \mathbb{R})$ is a Banach space with respect to the norm $\|F\| = \sup\{|F(f)| : f \in L^p(X), \|f\|_p \le 1\}$

<u>Proof</u>: Thm 276.1 in Book.

Thm: If $1 \le p \le \infty$,

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

$$g \in L^{p'}(X),$$

Then

$$F(f) = \int_X fg \, d\mu.$$

defines a bounded linear functional on $L^p(X)$ with

$$\|F\| = \|g\|_{p'}$$

Proof: Clearly F is linear.

$$|F(f)| = \int_X fg \, d\mu \le \int_X |fg| \, d\mu \le \|f\|_p \|g\|_{p'}$$

$$\Rightarrow \quad |F(f)| \le \|g\|_{p'} \quad \forall f, \ \|f\|_p \le 1$$

$$\Rightarrow \quad \|F\| \le \|g\|_{p'}$$

Thm 165.2 $\Rightarrow \|F\| = \sup\left\{ \int fg : \|f\|_p \le 1 \right\} = \|g\|_{p'}$

Thm: $1 < p < \infty$

F bounded linear functional on $L^p(X)$

$\Rightarrow \exists\, g \in L^{p'}(X)$, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ s.t

$$F(f) = \int_X fg \, d\mu \quad \forall f \in L^p(X).$$

Moreover. $\|g\|_{p'} = \|F\|$, g unique

If $p=1$, the same conclusion holds under the additional assumption that μ is σ-finite.

6.5 + 6.6 Signed Measures and the Radon-Nikodym Theorem.

<u>Def:</u> An extended real-valued function ν defined on a σ-algebra $\mathcal{M}$ is a <u>signed measure</u> if:

(i) ν assumes at most one of the values $+\infty, -\infty$,

(ii) $\nu(\phi) = 0$

(iii) If $\{E_k\}_{k=1}^{\infty}$ is a disjoint sequence of measurable sets then

$$\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k),$$

where the series either converges absolutely or diverges to $\pm\infty$.

<u>Def:</u> $(X, \mathcal{M}, \mu)$, measure μ, ν signed measure defined on $\mathcal{M}$

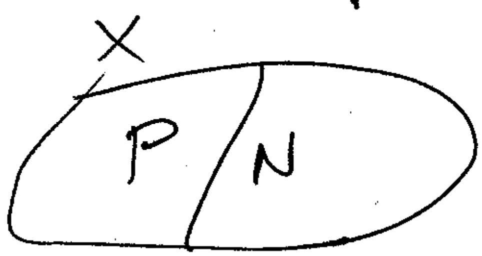
We say

$$\nu << \mu$$

"ν is absolutely continuous with respect to μ if:

$$\mu(E)=0 \Rightarrow \nu(E)=0$$

Thm: Hahn Decomposition:



$X = P \cup N$

$P \cap N = \emptyset$

$\nu(E) \geq 0, \quad \forall E \subset P$

$\nu(E) \leq 0, \quad \forall E \subset N$

Note: The Hahn Decomposition is not unique if ν has a non-empty null set. A set $C \in \mathcal{M}$ is a null set for ν if $\nu(E)=0$ $\forall E \subset C$, $E \in \mathcal{M}$.

Def: P is a positive set, N is a negative set.

Def: μ_1, μ_2 measures in $(X, \mathcal{M})$.

μ_1, μ_2 are mutually singular;

$$\mu_1 \perp \mu_2$$

if $\exists E \in \mathcal{M}$ s.t $\mu_1(E) = 0 = \mu_2(X \setminus E)$

Thm: Jordan Decomposition.

ν signed measure on $\mathcal{M}$.

Then there exist measures ν^+, ν^+, at least one of which is finite, s.t:

$$\nu(E) = \nu^+(E) - \nu^-(E), \ \forall E \in \mathcal{M}$$

Proof: $X = P \cup N$

$$\nu^+(E) := \nu(E \cap P)$$
$$\nu^-(E) := -\nu(E \cap N), \ \forall E \in \mathcal{M}.$$

Definition; The total variation of ν is defined as $|\nu| = \nu^+ + \nu^-$.

Note: $\nu \ll \mu \iff |\nu| \ll \mu \iff \nu^+ \ll \mu \ \& \ \nu^- \ll \mu$

Thm: If $\nu \ll \mu$ then $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow |\nu(E)| < \varepsilon$ as in Lemma 1 in Lesson 27.

Proof: Proceed by contradiction as in Lemma 1 in Lesson 27.

Thm: (Radon-Nikodym).

$(X, \mathcal{M}, \mu)$ σ-finite measure space.

ν σ-finite signed measure on $\mathcal{M}$

$\nu \ll \mu$

$\Rightarrow \exists f$ measurable s.t f^+ or f^- integrable, and $\nu(E) = \int_E f \, d\mu$, $\forall E \in \mathcal{M}$

<u>Notation</u>: $f := \frac{d\nu}{d\mu}$

f is called the Radon-Nikodym derivative of ν with respect to μ.

Proof of the Riesz Representation Theorem.

Assume $\mu(X)<\infty$,

Def.

$$\nu(E)=F(\chi_E), \qquad E\in\mathcal{M}$$

Claim: ν is a signed measure.

Let $\{E_k\}$, $E_k\in\mathcal{M}$ disjoints

We need to prove:

$$\nu\Big(\bigcup_{k=1}^{\infty}E_k\Big)=\sum_{k=1}^{\infty}\nu(E_k)$$

Since μ is a measure:

$$\mu\Big(\bigcup_{k=1}^{\infty}E_k\Big)=\sum_{k=1}^{\infty}\mu(E_k).$$

$$\therefore\quad \mu\Big(\bigcup_{k=N+1}^{\infty}\mu(E_k)\Big)=\sum_{k=N+1}^{\infty}\mu(E_k)\to 0$$

as $N\to\infty$

Let $s_N = \sum_{k=1}^{N} \nu(E_k)$

Need to show

$$s_N \to \nu(E), \quad \text{as } N \to \infty.$$

$$E = \bigcup_{k=1}^{\infty} E_k.$$

$$\left| \nu(E) - \sum_{k=1}^{N} \nu(E_k) \right| =$$

$$= \left| F\left(\chi_E - \sum_{k=1}^{N} \chi_{E_k}\right) \right|$$

$$= \left| F\left(\sum_{k=N+1}^{\infty} \chi_{E_k}\right) \right|, \quad \text{since } \chi_E = \sum_{k=1}^{\infty} \chi_{E_k}$$

$$\leq \|F\| \left\| \sum_{k=N+1}^{\infty} \chi_{E_k} \right\|_p$$

$$= \|F\| \left(\int_X \sum_{k=N+1}^{\infty} \chi_{E_k} \right)^{1/p}$$

$$= \|F\| \left(\sum_{k=N+1}^{\infty} \mu(E_k) \right)^{1/p}$$

$$= \|F\| \left(\mu\left(\bigcup_{k=N+1}^{\infty} E_k \right) \right)^{1/p}$$

$$\to 0 \quad \text{as } N \to \infty.$$

$$\therefore \quad \nu(E) = \sum_{K=1}^{\infty} \nu(E_K) \quad (*)$$

$(*)$ is true for every reanangment,

Since, for every reanangment:

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_{K_n})$$

$\therefore$ ν is a signed measure

<u>Claim</u>: $\nu << \mu$

$$\begin{aligned} |\nu(E)| &= |F(\chi_E)| \\ &\leq \|F\| \, \|\chi_E\|_p \\ &= \|F\| \left(\int_X \chi_E^p \right)^{1/p} \\ &= \|F\| \cdot (\mu(E))^{1/p} \end{aligned}$$

$$\therefore \quad \mu(E) = 0 \implies \nu(E) = 0.$$