Lesson 29

29.1

Proof of Riesz Representation Theorem, (RRT). Continuation:

Given $F \in \mathcal{B}(L^p(X, \mathcal{M}, \mu), \mathbb{R})$, a bounded linear functional from $L^p(X)$ into $\mathbb{R}$. We consider

$$\nu(E) = F(\chi_E), \quad \forall E \in \mathcal{M}.$$

We are first considering the case:

$$\underline{\mu(X) < \infty.}$$

$\mu(X) < \infty \Rightarrow \chi_E \in L^p(X)$ and $\nu(E)$ is well defined. We have also proved that ν is a measure and

$$\nu << \mu.$$

Thus, the Radon-Nikodym Theorem yields g measurable, g^+ or g^- integrable such that:

$$\nu(E) = \int_E g \, d\mu \quad \forall E \in \mathcal{M} \rightarrow (1)$$

Actually, g is integrable since $\mu(X)<\infty$ implies that $\chi_X \in L^p$ and hence $F(\chi_X)<\infty$. Thus,

$$F(\chi_X) = \nu(X) = \int_X g \, d\mu < \infty.$$

Therefore, $g \in L^1(X)$. Note from (1):

$$F(\chi_E) = \nu(E) = \int_E g \, d\mu = \int_X \chi_E g \, d\mu \quad \forall E \in \mathcal{M}$$

Hence:

$$\boxed{F(\chi_E) = \int_X \chi_E g \, d\mu \quad \forall E \in \mathcal{M}} \rightarrow (1.1)$$

Since (1.1) holds for characteristic functions then it also holds for simple functions, since a simple function is of the form $\sum_{i=1}^{n} a_i \chi_{E_i}$.

Thus:

$$(2) \rightarrow \boxed{F(f) = \int_X f g \, d\mu, \quad \text{for all simple functions}}$$

But we want (2) to hold for every $f \in L^p(X)$. Moreover, the statement of the RRT requires $g \in L^{p'}(X)$.

We proof this in the next 5 steps.

<u>Step 1</u>: Assume $\mu(X) < \infty$ and $p = 1$.

We proceed to show that $g \in L^\infty(X)$.
Assume that $\|g^+\|_\infty > \|F\|$. Let $M > 0$ such that:

$$\|g^+\|_\infty > M > \|F\|$$

and set:

$$E_M := \{x : g(x) > M\} = \{x : g^+(x) > M\}$$

Note that $\mu(E_M) > 0$ since otherwise we would have $\|g^+_\infty\| \leq M$. Then

$$M\mu(E_M) \leq \int \chi_{E_M} g \, d\mu = F(\chi_{E_M}) \leq \|F\| \, \|\chi_{E_M}\|_p \leq \|F\| \mu(E_M),$$

$$\Rightarrow \quad M\mu(E_M) \leq \|F\| \mu(E_M) \qquad (p=1)$$

$\Rightarrow \quad \|F\| \geq M$, which is a contradiction.

We conclude that:

$$\|g^+\|_\infty \leq M$$

In the same way we show $\|g^-\|_\infty \leq \|F\|$.

$$\Rightarrow \quad \|g\|_\infty \leq \|F\| \Rightarrow \boxed{g \in L^\infty(X)}$$

Since we are considering the case $p=1$, then, for any $f \in L^1(X)$, from Theorem 141.1:

$\exists\ f_k^+, f_k^-$ simple functions, $f_k^+ \geq 0$, $f_k^- \geq 0$ such that

$$f_k^+ \uparrow f^+ \quad , \quad f_k^- \uparrow f^-$$

Define:

$$f_k := f_k^+ - f_k^-$$

Then

$$f_k(x) = f_k^+(x) - f_k^-(x) \longrightarrow f^+(x) - f^-(x) = f(x)$$

for every x;

That is:

$$\boxed{f_k \longrightarrow f \text{ pointwise}}$$

Note:

$$|f_k| = |f_k^+ - f_k^-| \leq f_k^+ + f_k^- \leq f^+ + f^- = |f|$$

$\therefore\ |f_k| \leq |f| \in L^1(X)$, so the LDCT implies that $\|f_k - f\|_1 \longrightarrow 0$. Thus:

$$\left|F(f) - \int_X fg\,d\mu\right| = \left|F(f) - F(f_k) + F(f_k) - \int_X fg\,d\mu\right|$$
$$= \left|F(f - f_k) + \int_X f_k g\,d\mu - \int_X fg\,d\mu\right|, \text{ by } (2)$$

$$= \left| F(f-f_k) + \int_X (f_k - f) g \, d\mu \right|$$

$$\le |F(f-f_k)| + \int_X |f_k - f| |g| \, d\mu$$

$$\le \|F\| \, \|f - f_k\|_1 + \|f_k - f\|_1 \|g\|_\infty \quad ; \text{ by Holder's inequality}$$

Letting $k \to \infty$ we obtain, since $\|f_k - f\|_1 \to 0$;

$$\left| F(f) - \int_X fg \, d\mu \right| = 0$$

$$\therefore \quad F(f) = \int_X fg \, d\mu \quad \forall f \in L^1(X)$$

Finally, by Theorem 163.1 we have:

$$\|F\| = \sup \{ |F(f)| : f \in L^1(X), \ \|f\|_1 \le 1 \}$$

$$= \sup \left\{ \int_X fg : f \in L^1(X), \ \|f\|_1 \le 1 \right\}$$

$$= \|g\|_\infty .$$

Step 2: Assume $\mu(X) < \infty$ and $1 < p < \infty$.

Proceeding as above we construct a sequence of simple functions $\{f_k\}$ such that, for $f \in L^p(X)$:

$$f_k \to f \text{ pointwise}, \ |f_k| \le |f| \in L^p(X) \quad \to (3)$$

Note that (3) implies:

$|f_k - f|^p \to 0$ pointwise and $|f_k - f|^p \le 2^{p-1}(|f_k|^p + |f|^p)$

$$\le 2^p |f|^p \in L^1(X)$$

Thus, we can apply again the LDCT to obtain:

$$\int_X |f_k - f|^p \to 0.$$

Then; as in the case $p=1$:

$$(4) \quad \left|F(f) - \int_X fg\,d\mu\right| \le |F(f-f_k)| + \left|\int (f_k - f)g\,d\mu\right|$$
$$\le \|F\|\,\|f-f_k\|_p + \|f_k - f\|_p\,\|g\|_{p'},$$

However, we need to show $\|g\|_{p'} < \infty$. In order to see this, let $\{h_k\}$ (by Theorem 141.1) be an increasing sequence of nonnegative simple functions such that

$$h_k \uparrow |g|.$$

Define:

$$g_k = h_k^{p'-1}\,\mathrm{sign}(g). \quad \text{Then}$$

$$\|h_k\|_{p'}^{p'} = \int_X h_k^{p'}\,d\mu = \int_X h_k^{p'-1}\,h_k\,d\mu$$
$$\le \int_X h_k^{p'-1}\,|g|\,d\mu$$
$$= \int_X h_k^{p'-1}\,\mathrm{sign}(g)\,g\,d\mu$$
$$= F(g_k)\,; \quad \text{by } (2)$$

$$\leq \|F\| \, \|g_k\|_p$$

$$= \|F\| \left(\int_X (h_k^{p'-1})^p \right)^{1/p}$$

$$= \|F\| \left(\int_X h_k^{p'} \right)^{1/p} \quad ; \quad \frac{1}{p} + \frac{1}{p'} = 1$$

$$p' + p = pp'$$

$$pp' - p = p'$$

$$= \|F\| \left(\|h_k\|_{p'}^{p'} \right)^{1/p}$$

$$= \|F\| \quad \|h_k\|_{p'}^{p'/p} \quad \rightarrow (5)$$

We need to prove that $g \in L^{p'}(X)$. If $g \equiv 0$ we are done. We assume $\|g\|_{p'} > 0$ and hence $\|h_k\|_{p'} > 0$ for large k (From Egorov's Thm).

Then:

$$(5) \Rightarrow \|h_k\|_{p'}^{p' - p'/p} \leq \|F\|$$

$$\Rightarrow \quad \|h_k\|_{p'} \leq \|F\| , \quad ; \quad \frac{1}{p} + \frac{1}{p'} = 1$$

$$\frac{p'}{p} + 1 = p'$$

$$p' - \frac{p'}{p} = 1$$

for large k.

Using now Fatou's Lemma:

$$\int_X \liminf_{k\to\infty} h_k^{p'} \le \liminf_{k\to\infty} \int_X h_k^{p'}$$

$$\therefore \int_X |g|^{p'} \le \liminf_{k\to\infty} \|h_k\|_{p'}^{p'}$$

$$\le \|F\|^{p'}$$

$$\therefore \left(\int_X |g|^{p'}\right)^{1/p'} \le \|F\|$$

$$\therefore \boxed{\|g\|_{p'} \le \|F\|.} \Rightarrow \boxed{g \in L^{p'}(X)}.$$

Going back now to (4) we obtain; since $\|f_k - f\|_p \to 0$ and $\|g\|_{p'} < \infty$ that

$$\left| F(f) - \int_X fg \, d\mu \right| = 0$$

That is:

$$\boxed{F(f) = \int_X fg \, d\mu \quad \forall f \in L^p(X)}$$

Step 3 : Assume μ is σ-finite and $1 \le p < \infty$

(Sketch of Proof)

Let $Y \in \mathcal{M}$, σ-finite.

$$\Rightarrow \quad Y = \bigcup_{k=1}^{\infty} Y_k, \quad \mu(Y_k) < \infty$$

Since $\mu(Y_k) < \infty$, Step 1 + Step 2 + Excercise 6.7 below implies:

$$\forall k, \ \exists g_k, \ \|g_k \chi_{Y_k}\|_{p'} \le \|F\| \quad \text{and}$$

$$\boxed{F(f\chi_{Y_k}) = \int_X f\chi_{Y_k} g_k \, d\mu, \ \forall f \in L^p(X).}$$ We may assume $g_k \equiv 0$ on $Y \setminus Y_k$.

For $m > k$

$$F(f\chi_{Y_k}) = \int_X f g_m \chi_{Y_k} \, d\mu \quad \forall f \in L^p(X)$$

$$\Rightarrow \int_X f\,(g_k - g_m \chi_{Y_k})\, d\mu = 0 \quad \forall f \in L^p(X)$$

$$\therefore \ \boxed{g_k = g_m \ \ \mu\text{-a.e. } x \text{ on } Y_k, \text{ by Thm 163.1}}$$

(6.7): Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $Y \in \mathcal{M}$. Set:

$$\mu_Y(E) := \mu(E \cap Y) \quad \forall E \in \mathcal{M}.$$

Show that μ_Y is a measure on $(X, \mathcal{M})$ &

$$\int_X g \, d\mu_Y = \int_X g \chi_Y \, d\mu \quad \forall g \text{ measurable on } X.$$

$\Rightarrow$ $\{g_k\}$ converges μ-a.e. to a measurable function g.

We can now proceed as in previous steps to conclude that $g \in L^{p'}(X)$ and:

$$g_k = g \quad \mu\text{-a.e. on } Y_k$$

and

$$\boxed{F(f\chi_Y) = \int_Y fg\,d\mu \quad \forall f \in L^p(X)}$$

If μ is σ-finite, let $Y = X$ and deduce:

$$\boxed{F(f) = \int_X fg\,d\mu \quad \forall f \in L^p(X)},$$

Where $g \in L^{p'}(X)$ and:

$$\|F\| = \sup\{|F(f)| : \|f\|_p = 1\} = \|g\|_{p'}.$$

Step 4 : μ is not σ-finite and $1<p<\infty$.

(Sketch of Proof).

$\forall K \quad \exists h_K \in L^p(X)$, $\|h_K\|_p \le 1$ and

$$\|F\| - \frac{1}{K} < |F(h_K)| \le \|F\|$$

Set:

$$Y = \bigcup_{K=1}^{\infty} \{x : h_K(x) \neq 0\}$$

Y is measurable, σ-finite subset of X.

Step 3 $\Rightarrow$ $\exists g \in L^{p'}(X)$, $g \equiv 0$ μ-a.e on $X - Y$ on $X \setminus Y$ and

$$\boxed{F(f\chi_Y) = \int_Y fg \, d\mu \quad \forall f \in L^p(X)}$$

Then we need to show that

$$F(f) = F(f\chi_Y) \quad \forall f \in L^p(X)$$

and $$\|g\|_{p'} = \|F\|.$$

Step 5 : Suppose $\exists \tilde{g} \in L^{p'}(X)$ such that $\int_X f(g - \tilde{g})\, d\mu = 0 \quad \forall f \in L^p(X)$. Then Theorem 163.1 implies that $\|g - \tilde{g}\|_{p'} = 0 \Rightarrow g = \tilde{g}$ μ-a.e $\Rightarrow g$ is unique