# Lesson 3

<u>Thm</u>: Let $\varphi$ be an outer measure on an arbitrary set $X$. Then

(i) $\varphi(E) = 0 \Rightarrow E$ is $\varphi$-measurable

(ii) $\emptyset$ and $X$ are $\varphi$-measurable

(iii) If $E_1, E_2$ are $\varphi$-measurable $\Rightarrow E_1 \setminus E_2$ is $\varphi$-measurable

(iv) Let $\{E_i\}$ be a countable collection of disjoint $\varphi$-measurable sets, then $\bigcup_{i=1}^{\infty} E_i$ is $\varphi$-measurable and

$$\varphi\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \varphi(E_i)$$

More generally, if $A \subset X$ is an arbitrary set, then:

$$\varphi(A) = \sum_{i=1}^{\infty} \varphi(A \cap E_i) + \varphi(A \cap S^c),$$

where

$$S = \bigcup_{i=1}^{\infty} E_i$$

Proof

(i) Assume $\varphi(E) = 0$

Let $P, Q$, $P \subset E$, $Q \subset E^c$

By subadditivity:

$$\varphi(P \cup Q) \leq \varphi(P) + \varphi(Q)$$

We only need to show the reverse inequality.

Since $P \subset E$, by monotonicity:

$$\varphi(P) \leq \varphi(E) = 0$$

$$\therefore \varphi(P) = 0 \quad \longrightarrow (1)$$

Since $Q \subset P \cup Q$, by monotonicity,

$$\varphi(Q) \leq \varphi(P \cup Q) \longrightarrow (2)$$

From (1) & (2):

$$\varphi(P) + \varphi(Q) \leq \varphi(P \cup Q). \quad \blacksquare$$

(ii) Let $P \subset \phi$, $Q \subset X$

$\Rightarrow \varphi(P) \leq \varphi(\emptyset) = 0$.

Since $Q \subset P \cup Q$, then $\varphi(Q) \leq \varphi(P \cup Q)$

$\therefore \varphi(P) + \varphi(Q) \leq \varphi(P \cup Q) \quad \blacksquare$

Let $P \subset X$, $Q \subset X^c = \emptyset$

$\therefore\ \varphi(Q) = 0$

Since $P \subset P \cup Q$, then $\varphi(P) \leq \varphi(P \cup Q)$

Hence:

$\varphi(Q) + \varphi(P) \leq \varphi(P \cup Q)$. ∎

(iii) Let $E_1, E_2$ $\varphi$-measurable

Let $P \subset E_1 \setminus E_2$, $Q \subset (E_1 \setminus E_2)^c$

<u>Note:</u> $(E_1 \setminus E_2)^c = (E_1 \cap E_2^c)^c = E_1^c \cup E_2$

Then:

$$\varphi(P) + \varphi(Q) = \varphi(P) + \underbrace{\varphi(Q \cap E_2) + \varphi(Q \cap E_2^c)}_{\varphi(Q)\text{; since } E_2 \text{ is } \varphi\text{-measurable}}$$

$$= \underbrace{\varphi(P) + \varphi(Q \cap E_2^c)}_{\varphi(P \cup (Q \setminus E_2))\text{; since } E_1 \text{ is } \varphi\text{-measurable}} + \varphi(Q \cap E_2)$$

because $[P \cup (Q \setminus E_2)] \cap E_1 = (P \cap E_1) \cup [(Q \setminus E_2) \cap E_1]$
$= P \cup \emptyset = P$

$[P \cup (Q \setminus E_2)] \cap E_1^c = (P \cap E_1^c) \cup [(Q \setminus E_2) \cap E_1^c]$
$= \emptyset \cup (Q \setminus E_2) = Q \setminus E_2$

Hence:

$$\varphi(P)+\varphi(Q) = \varphi(P\cup(Q\setminus E_2)) + \varphi(Q\cap E_2)$$
$$= \varphi(P\cup(Q\cap E_2^c)) + \varphi(Q\cap E_2)$$

Since $Q\cap E_2 \subset E_2$ and $P\cup(Q\cap E_2^c) \subset E_2^c$, the measurability of $E_2$ yields:

$$\varphi(P)+\varphi(Q) = \varphi\left[(P\cup(Q\cap E_2^c)) \cup (Q\cap E_2)\right]$$
$$= \varphi\left[P \cup \underbrace{(Q\cap E_2^c) \cup (Q\cap E_2)}_{=Q}\right]$$
$$= \varphi(P\cup Q).$$

We conclude:

$$\varphi(P\cup Q) = \varphi(P)+\varphi(Q). \quad \blacksquare$$

(iv) $\{E_i\}$ countable collection of disjoint $\varphi$-measurable sets.

Define:

$$S_k := \bigcup_{i=1}^{k} E_i$$

Let $A \subset X$ be an arbitrary set.

We proceed by induction:

- The result is true for $k=1$, since: $E_1$ is $\varphi$-meas. and $\varphi(A) \geq \varphi(A \cap E_1) + \varphi(A \cap E_1^c)$

- We assume that the result is true for $k$; that is, we assume: $S_k$ is $\varphi$-meas. & $\varphi(A) \geq \sum_{i=1}^{k} \varphi(A \cap E_i) + \varphi(A \cap S_k^c)$

- We need to show that the result is true for $k+1$:

We have:

$$\varphi(A) = \varphi(A \cap E_{k+1}) + \varphi(A \cap E_{k+1}^c); \text{ Since } E_{k+1} \text{ is } \varphi\text{-measurable}$$

$$= \varphi(A \cap E_{k+1}) + \varphi(A \cap E_{k+1}^c \cap S_k) + \varphi(A \cap E_{k+1}^c \cap S_k^c); \text{ Since } S_k \text{ is } \varphi\text{-meas.}$$

$\therefore$

$$\varphi(A) = \varphi(A \cap E_{k+1}) + \varphi(A \cap S_k) + \varphi\left(A \cap \left(\bigcap_{i=1}^{k+1} E_i^c\right)\right); \text{ Since } S_k \subset E_{k+1}^c$$

$$= \varphi(A \cap E_{k+1}) + \varphi(A \cap S_k) + \varphi(A \cap S_{k+1}^c); \text{ Since } S_{k+1}^c = \left(\bigcup_{i=1}^{k+1} E_i\right)^c = \bigcap_{i=1}^{k+1} E_i^c$$

$$\geq \varphi(A \cap E_{k+1}) + \varphi(A \cap S_{k+1}^c) + \underbrace{\sum_{i=1}^{k} \varphi(A \cap S_k \cap E_i) + \varphi((A \cap S_k) \cap S_k^c)}_{\text{From the hypothesis of induction}}$$

$$= \varphi(A \cap E_{k+1}) + \sum_{i=1}^{k} \varphi(A \cap E_i) + \varphi(A \cap S_{k+1}^c)$$

$$\boxed{\varphi(A) \geq \sum_{i=1}^{k+1} \varphi(A \cap E_i) + \varphi(A \cap S_{k+1}^c)} \quad (3)$$

(3) $\Rightarrow$ Result is true for $K+1$.

The subadditivity property and (3) give:

$$\boxed{\varphi(A) \geq \varphi(A \cap S_{K+1}) + \varphi(A \cap S_{K+1}^c)} \quad (4)$$

(4) says that:

$$\boxed{S_{K+1} \text{ is } \varphi\text{-measurable.}}$$

Since $S^c \subset S_K^c, \forall K$, then from (3) and monotonicity:

$$\varphi(A) \geq \sum_{i=1}^{K+1} \varphi(A \cap E_i) + \varphi(A \cap S^c), \forall K$$

$$\therefore \boxed{\varphi(A) \geq \sum_{i=1}^{\infty} \varphi(A \cap E_i) + \varphi(A \cap S^c)} \quad (5)$$

By (5) and subadditivity:

$$\boxed{\varphi(A) \geq \varphi(A \cap S) + \varphi(A \cap S^c)} \quad (6)$$

By (6) we conclude:

$$\boxed{S \text{ is } \varphi\text{-measurable}}$$

Hence:

$$\begin{aligned}\varphi(A) &= \varphi(A\cap S) + \varphi(A\cap S^c)\\ &= \varphi\Big(\bigcup_{i=1}^{\infty}(A\cap E_i)\Big) + \varphi(A\cap S^c)\\ &\le \sum_{i=1}^{\infty}\varphi(A\cap E_i) + \varphi(A\cap S^c); \quad \text{by subadditivity}\end{aligned}$$

Hence, (5) yields:

$$\boxed{\varphi(A) = \sum_{i=1}^{\infty}\varphi(A\cap E_i) + \varphi(A\cap S^c)} \quad (7)$$

Putting $A := \bigcup_{i=1}^{\infty} E_i$ in (7) yields:

$$\varphi\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty}\varphi\Big[\Big(\bigcup_{i=1}^{\infty} E_i\Big)\cap E_i\Big] + \varphi(S\cap S^c)$$

$$\Rightarrow \boxed{\varphi\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty}\varphi(E_i)} \quad \blacksquare$$