Lesson 30

Product measures & Fubini's Theorem.

Let μ and ν measures on X defined on σ-algebras $\mathcal{M}_X$ and $\mathcal{M}_Y$ repectively

We want to define a measure in $X \times Y$.

Def: For each $S \subset X \times Y$ define:

$$\varphi(S) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A_j)\,\nu(B_j) \right\},$$

where the infimum is taken over all sequences $\{A_j \times B_j\}$ such that:

$$S \subset \bigcup_{j=1}^{\infty} A_j \times B_j, \quad A_j \in \mathcal{M}_X, \; B_j \in \mathcal{M}_Y.$$

Thm: φ is an outer measure on $X \times Y$.

Proof:

(i) $\varphi \geq 0$

(ii) $\varphi(\emptyset) = 0$.

Need to show that φ is countably subadditive.

Consider

$$\bigcup_{k=1}^{\infty} S_k \quad , \quad S_k \subset X \times Y$$

WLOG assume $\varphi(S_k) < \infty$, $\forall k$ (for otherwise clearly $\varphi(\bigcup_{k=1}^{\infty} S_k) \leq \sum_{k=1}^{\infty} \varphi(S_k)$)

Fix $\varepsilon > 0$.

By definition of infimum $\Rightarrow$

$\exists \{A_j^k \times B_j^k\}_{j=1}^{\infty}$, $A_j^k \in \mathcal{M}_X$, $B_j^k \in \mathcal{M}_Y$, $\forall k$

s.t

$$\sum_{j=1}^{\infty} \mu(A_j^k)\, \nu(B_j^k) < \varphi(S_k) + \frac{\varepsilon}{2^k}$$

$$\therefore \varphi\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu(A_j^k)\, \nu(B_j^k)$$

$$\leq \sum_{k=1}^{\infty} \left(\varphi(S_k) + \frac{\varepsilon}{2^k} \right)$$

$$= \sum_{k=1}^{\infty} \varphi(S_k) + \varepsilon .$$

ε arbitrary $\Rightarrow$

$$\varphi\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \varphi(S_k)$$

We need to prove monotonicity.

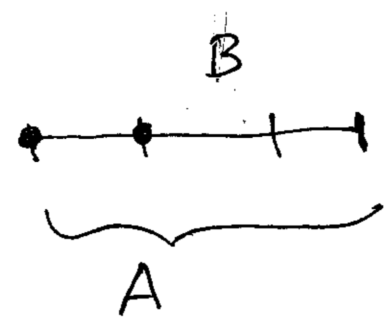
Let

$$S_1 \subset S_2.$$

A cover of S_2 is also a cover of S_1;

ie:

$$\{\text{Covers of } S_2\} \subset \{\text{Covers of } S_1\}$$



Note: if $B \subset A$ then $\inf A \le \inf B$.

Hence

$$\varphi(S_1) \le \varphi(S_2). \quad \blacksquare$$

<u>Note</u>: The outer measure φ generates the measure space:

$$(X \times Y, \mathcal{M}_{X\times Y}, \mu \times \nu),$$

where $\mathcal{M}_{X\times Y}$ is the σ-algebra of φ-measurable sets

and $$\mu\times\nu = \varphi|_{\mathcal{M}_{X\times Y}}.$$

$\mu \times \nu$ is called a product measure.

Fubini's Theorem

Suppose $(X, \mathcal{M}_X, \mu)$ & $(Y, \mathcal{M}_Y, \nu)$ are complete measure spaces. 30.4

(i) If $A \in \mathcal{M}_X$, $B \in \mathcal{M}_Y \Rightarrow A \times B \in \mathcal{M}_{X \times Y}$ and

$$(\mu \times \nu)(A \times B) = \mu(A)\,\nu(B).$$

(ii) If $S \in \mathcal{M}_{X\times Y}$ and S is σ-finite with respect to $\mu \times \nu$:
$\Rightarrow$

$S_y = \{x : (x,y) \in S\} \in \mathcal{M}_X$, ν-a.e. $y \in Y$
$S_x = \{y : (x,y) \in S\} \in \mathcal{M}_Y$, μ-a.e. $x \in X$
$y \mapsto \mu(S_y)$ is $\mathcal{M}_Y$-measurable, $x \mapsto \nu(S_x)$ is $\mathcal{M}_X$-meas.

$$(\mu \times \nu)(S) = \int_X \nu(S_x)\, d\mu(x) = \int_X \left[\int_Y \chi_S(x,y)\, d\nu(y)\right] d\mu(x)$$
$$= \int_Y \mu(S_y)\, d\nu(y) = \int_Y \left[\int_X \chi_S(x,y)\, d\mu(x)\right] d\nu(y).$$

(iii) If $f \in L^1(X \times Y, \mathcal{M}_{X\times Y}, \mu \times \nu)$ then

$y \mapsto f(x,y)$ is ν-integrable for μ-a.e. $x \in X$
$x \mapsto f(x,y)$ is μ-integrable for ν-a.e. $y \in Y$
$x \mapsto \int_Y f(x,y)\, d\nu(y)$ is μ-integrable
$y \mapsto \int_X f(x,y)\, d\mu(x)$ is ν-integrable.

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_X \left[\int_Y f(x,y)\, d\nu(y)\right] d\mu(x) = \int_Y \left[\int_X f(x,y)\, d\mu(x)\right] d\nu(y)$$

The n- dimensional Lebesgue measure on $\mathbb{R}^n$ is a product of lower dimensional Lebesgue measures

We have:

<u>Thm</u>: For each pair of positive integers n and m:

$$\lambda_{n+m} = \lambda_n \times \lambda_m .$$

That is:

$$\int_{\mathbb{R}^{n+m}} f \, d\lambda_{n+m} = \int_{\mathbb{R}^n \times \mathbb{R}^m} f \, d(\lambda_n \times \lambda_m)$$

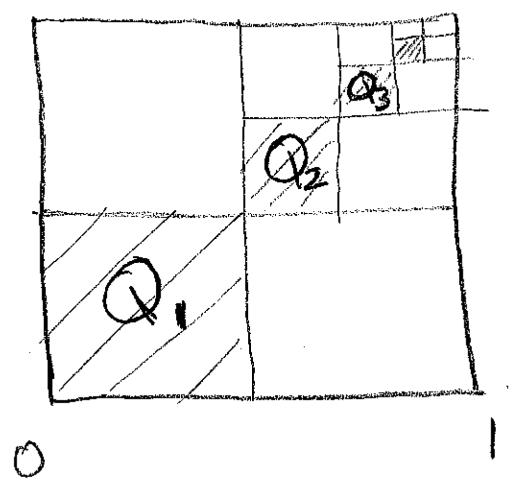
In this, Fibini's Theorem reduces to:

$$\int_{\mathbb{R}^{n+m}} f \, d\lambda_{n+m} = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^m} f(x,y) \, d\lambda_m(y) \right] d\lambda_n(x)$$
$$= \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^n} f(x,y) \, d\lambda_n(x) \right] d\lambda_m(x)$$

In particular, for example:

$$\int_{\mathbb{R}^2} f(x,y) \, d\lambda_2 = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y) \, d\lambda(y) \right] d\lambda(x)$$
$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y) \, d\lambda(x) \right] d\lambda(y) .$$

Ex: The Hypothesis $f \in L^1(X \times Y)$ is necessary: We construct a function f as follows:



$$Q = [0,1] \times [0,1]$$

We have a sequence $\{Q_K\}$. Each Q_K is partitioned as follows:

$$Q_K = \begin{array}{|c|c|}\hline Q_K^{(2)} \;\; - & Q_K^{(1)} \;\; + \\ \hline Q_K^{(3)} \;\; + & Q_K^{(4)} \;\; - \\ \hline \end{array}$$

Define f on Q as follows:

$$f(x) = \begin{cases} 0, & x \notin Q_K, \ \forall K \\ \dfrac{1}{\lambda_2(Q_K)}, & x \in Q_K^{(1)} \cup Q_K^{(3)} \\ -\dfrac{1}{\lambda_2(Q_K)}, & x \in Q_K^{(2)} \cup Q_K^{(4)} \end{cases}$$

Note:

$$\int_{Q_K} |f|\, d\lambda_2 = \int_{Q_K^{(1)}} \frac{1}{\lambda_2(Q_K)}\, d\lambda_2$$

$$+ \int_{Q_K^{(3)}} \frac{1}{\lambda_2(Q_K)}\, d\lambda_2$$

$$+ \int_{Q_K^{(2)}} \frac{1}{\lambda_2(Q_K)}\, d\lambda_2$$

$$+ \int_{Q_K^{(4)}} \frac{1}{\lambda_2(Q_K)}\, d\lambda_2$$

$$= \frac{1}{\lambda_2(Q_K)} \left[\lambda_2(Q_K^{(1)}) + \lambda_2(Q_K^{(3)}) + \lambda_2(Q_K^{(2)}) + \lambda_2(Q_K^{(4)})\right]$$

$$= \frac{1}{\lambda_2(Q_K)} \lambda_2(Q_K) = 1$$

$$\therefore \int_Q |f|\, d\lambda_2 = \sum_K \int_{Q_K} |f| = \infty \Rightarrow f \notin L^1(Q)$$

and

$$\int_0^1 f(x,y)\, d\lambda_1(x) = 0 = \int_0^1 f(x,y)\, d\lambda_1(y)$$

This is an example where the iterated integrals exist but Fubini's Thm is not true because:

$$f \notin L^1(Q)$$

Corollary (Tonelli's Thm).
to Fubini

Thm -(Tonelli):

Let $f \geq 0$, f is $\mathcal{M}_{X \times Y}$- measurable and $\{(x,y) : f(x,y) \neq 0\}$ is σ-finite with respect to $\mu \times \nu$.

Then:

$y \mapsto f(x,y)$ is $\mathcal{M}_Y$- measurable for μ-a.e. X.

$x \mapsto f(x,y)$ is $\mathcal{M}_X$ - measurable for ν-a.e. Y.

$x \longmapsto \int_Y f(x,y)\, d\nu(y)$ is $\mathcal{M}_X$ - measurable

$y \longmapsto \int_X f(x,y)\, d\mu(x)$ is $\mathcal{M}_Y$ - measurable.

and:

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f(x,y)\, d\nu(y) \right] d\mu(x)$$

$$= \int_Y \left[\int_X f(x,y)\, d\mu(x) \right] d\nu(y)$$

in the sense that either both expressions are infinite or both are finite and equal.

Proof: From Thm 141.1 :

$\exists \{f_K\}$, $f_K \geq 0$ simple functions such that

$$f_K(x) \uparrow f(x), \quad \forall x.$$

From

$$f_K = \sum_{j=1}^{N(K)} a_{j,K} \chi_{S_{j,K}}$$

and since by hypothesis:

$\{(x,y) : f(x,y) \neq 0\}$ is σ-finite

then

$S_{j,K}$ is σ-finite with respect to $\mu \times \nu$, $\forall j, K$.

Fubini (ii) $\Rightarrow$ <u>Tonelli holds for each f_K</u>.

For each K, let N_K, $\mu(N_K) = 0$ s.t.

$y \mapsto f_K(x,y)$ is $\mathcal{M}_Y$-meas. $\forall x \notin N_K$

Let $N = \bigcup_{K=1}^{\infty} N_K \Rightarrow \mu(N) = 0$.

By Monotone Convergence Theorem

$$\boxed{\int_Y f(x,y)\, d\nu(y) = \lim_{K \to \infty} \int_Y f_K(x,y)\, d\nu(y) \quad (1) \qquad \forall x \notin N.}$$

because $y \mapsto f(x,y) = \lim_{K \to \infty} f_K(x,y)$ $\forall x \notin N$, and $y \mapsto f(x,y)$ is $\mathcal{M}_Y$-measurable.

Fubini $\Rightarrow$ (30.10)

$$\boxed{h_K(x) := \int_Y f_K(x,y)\, d\nu(y) \text{ is } \mathcal{M}_X\text{- measurable}} \quad (2)$$

<u>Note:</u>

$$0 \leq h_K \leq h_{K+1} \quad \to (3)$$

We use again the Monotone - Convergence Theorem:

$$\int_{X \times Y} f \, d(\mu \times \nu) = \lim_{K \to \infty} \int_{X \times Y} f_K(x,y)\, d(\mu \times \nu) \; ; \quad \text{by Monotone Convergence Theorem}$$

$$= \lim_{K \to \infty} \int_X \left[\int_Y f_K(x,y)\, d\nu(y) \right] d\mu(x); \quad \text{Because Tonelli holds for each } f_K.$$

$$= \lim_{K \to \infty} \int_X h_K(x)\, d\mu(x)$$

$$= \int_X \lim_{K \to \infty} h_K(x)\, d\mu(x) \; ; \quad \text{by Monotone Convergence Thm and (3)}$$

$$= \int_X \lim_{K \to \infty} \left[\int_Y f_K(x,y)\, d\nu(y) \right] d\mu(x) \; ; \quad \text{by (2)}$$

$$= \int_X \left[\int_Y f(x,y)\, d\nu(y) \right] d\mu(x) \; ; \quad \text{by (1)}$$