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Convolution

Def: Let $f, g : \mathbb{R}^n \to \mathbb{R}$ Lebesgue measurable functions. The convolution $f * g$ is defined as:

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$$

Remark: In this section we use the notation:

$$d\lambda = dx, dy, \text{ etc.}$$

However remember we are integrating with respect to Lebesgue measure (not Riemann) even though we use "dx".

Note: Let $g \geq 0$. Then

$$\int_{\mathbb{R}^n} g(x-y)\, dy = \int_{\mathbb{R}^n} g(y)\, dy .$$

This is true from the definition of integral and the fact that λ_n is invariant under translations. We can also perform a change of variables, but we haven't prove such a formula (See Chapter 11 in book).

<u>Lemma</u>: Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Define:

$$F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n} \text{ by}$$

$$F(x,y) = f(x-y)$$

Then F is λ_{2n}- measurable.

<u>Proof</u>: Def:

$$F_1 : \mathbb{R}^{2n} \to \mathbb{R}$$

$$F_1(x,y) = f(x)$$

F_1 is λ_{2n}- measurable because $\forall B \subset \mathbb{R}$ Borel: we have:

$$F_1^{-1}(B) = f^{-1}(B) \times \mathbb{R}^n$$

Def: $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by:

$$T(x,y) = (x-y,\ x+y)$$

Then: $T^{-1}(x,y) = \left(\frac{x+y}{2},\ \frac{y-x}{2}\right)$

The mean value Theorem $\Rightarrow$

$$|T(x_1,y_1) - T(x_2,y_2)| \le n^2[(x_1,y_1) - (x_2,y_2)]$$

$$|T^{-1}(x_1,y_1) - T^{-1}(x_2,y_2)| \le \frac{1}{2} n^2[(x_1,y_1) - (x_2,y_2)]$$

$$\forall (x_1,y_1),\ (x_2,y_2) \in \mathbb{R}^{2n}$$

Here we have used:

Let $A \subset \mathbb{R}^n$ be an interval and let $f : A \to \mathbb{R}^n$ be continuously differentiable. If $\exists M$ such that $\left| \frac{\partial f^i}{\partial x_j}(x) \right| \leq M$ for all x in the interior of A, then

$$| f(x) - f(y) | \leq n^2 M |x-y| \,, \ \forall x, y \in A.$$

$\therefore$ T & T^{-1} are Lipschitz functions in $\mathbb{R}^{2n}$.

<u>Claim</u>: $F = F_1 \circ T$ is λ_{2n}-measurable.

Let $B \subset \mathbb{R}$ a Borel set.

$$\Rightarrow \quad F^{-1}(B) = (F_1 \circ T)^{-1}(B)$$

$$= T^{-1}(F_1^{-1}(B))$$

$$= T^{-1}(E) \,, \quad E := F_1^{-1}(B) \text{ is Lebesgue measurable}$$

$$= T^{-1}(B_1 \cup N) \,; \quad B_1 \text{ is Borel and } \lambda_{2n}(N) = 0$$

$$= T^{-1}(B_1) \cup T^{-1}(N)$$

Note that

$T^{-1}(B_1)$ is Lebesgue measurable because T^{-1} is continuous. Also, since T^{-1} is Lipschitz, exercise 4.48 yields that

$$\lambda_n(T^{-1}(N)) = 0,$$

and hence Lebesgue measurable.

We conclude that $T^{-1}(E)$ is Lebesgue measurable

$\therefore$ $F^{-1}(B)$ is Lebesgue measurable

$\therefore$ F is Lebesgue measurable.

<u>Thm</u>: If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$ and $g \in L^1(\mathbb{R}^n)$, the $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f*g\|_p \le \|f\|_p \|g\|_1$$

<u>Proof</u>: Let $f, g \ge 0$

By previous Lemma the function:

$$\mathbb{R}^{2n} \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto f(y)\, g(x-y)$$

is Lebesgue measurable

<u>Case $p=1$</u>: In this case $f \in L^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Then:

$$\int_{\mathbb{R}^n} (f*g)(x)\,dx = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \right] dx$$

$$= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(y)\, g(x-y)\, dx \right] dy \quad ; \text{ By Tonelli's Theorem}$$

$$= \int_{\mathbb{R}^n} f(y) \left[\int_{\mathbb{R}^n} g(x-y)\, dx \right] dy$$

$$= \int_{\mathbb{R}^n} f(y) \left[\int_{\mathbb{R}^n} g(x)\, dx \right] dy$$

$$= \int_{\mathbb{R}^n} g(x)\, dx \int_{\mathbb{R}^n} f(y)\, dy$$

$$= \|f\|_1 \, \|g\|_1$$

$$\therefore \|f*g\|_1 = \|f\|_{L^1} \|g\|_{L^1}$$

<u>Case $p=\infty$</u>: $(f*g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$

$$\le \|f\|_\infty \int_{\mathbb{R}^n} g(x-y)\, dy$$

$$= \|f\|_\infty \, \|g\|_1$$

$$\therefore \|f*g\|_\infty \le \|f\|_\infty \|g\|_1$$

<u>Case $1<p<\infty$</u> :

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$$

$$= \int_{\mathbb{R}^n} f(y) \left(g(x-y) \right)^{1/p} \left(g(x-y) \right)^{1-1/p} dy$$

$$\leq \left(\int_{\mathbb{R}^n} f^p(y) g(x-y)\, dy \right)^{1/p} \left(\int_{\mathbb{R}^n} g(x-y)^{p' - \frac{p'}{p}} \right)^{1/p'} ; \text{ by Holder's inequality}$$

$$\frac{1}{p} + \frac{1}{p'} = 1 \Rightarrow \frac{p'}{p} + 1 = p'$$

$$\Rightarrow p' - \frac{p'}{p} = 1$$

$$= \left(f^p * g\,(x) \right)^{1/p} \|g\|_1^{1/p'}$$

$$\therefore \quad (f*g)(x) \leq \left[(f^p * g)(x) \right]^{1/p} \|g\|_1^{1/p'}$$

$$\therefore \quad (f*g)^p(x) \leq (f^p * g)(x) \|g\|_1^{p/p'} ; \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 + \frac{p}{p'} = p$$

$$\therefore \quad (f*g)^p(x) \leq (f^p * g)(x) \|g\|_1^{p-1}$$

$$\int_{\mathbb{R}^n} (f*g)^p(x)\, dx \leq \int_{\mathbb{R}^n} (f^p * g)(x) \|g\|_1^{p-1}$$

$$= \|f^p\|_1 \|g\|_1 \|g\|_1^{p-1} = \|f\|_p^p \|g\|_1^p$$

$$\left(\int_{\mathbb{R}^n} f*g^p(x)\,dx\right)^{1/p} \leq \|f\|_p \|g\|_1$$

$$\therefore \quad \|f*g\|_p \leq \|f\|_p \|g\|_1$$

Without the assumption $f, g \geq 0$ we have:

$$|f*g|(x) = \left| \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \right|$$

$$\leq \int_{\mathbb{R}^n} |f(y)|\,|g(x-y)|\,dy$$

$$= \int_{\mathbb{R}^n} |f|(y)\,|g|(x-y)\,dy$$

$$= |f| * |g|\,(x)$$

Since $|f|, |g| \geq 0$, <u>previous step</u> gives:

$$|f*g| \in L^p$$

and

$$\|f*g\|_p = \|\,|f*g|\,\|_p$$

$$\leq \|\,|f|\,\|_p \,\|\,|g|\,\|_1$$

$$= \|f\|_p \|g\|_1 .$$

<u>Def</u>: Fix $g \in L^1(\mathbb{R}^n)$. Let $T: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $T(f) = f*g$. Then T is a bounded linear mapping.

6.11 Distribution Functions

$(X, \mathcal{M}, \mu)$ complete σ-finite measure space.

f measurable function on X.

Let $t \in \mathbb{R}$. Def.

$$E_t = \{x : f(x) > t\} \in \mathcal{M}.$$

$$A_f(t) = \mu(E_t)$$

$t \longmapsto A_f(t)$ is called the distribution function.

<u>Thm</u>: $f \geq 0$ measurable, then

$$\int_X f \, d\mu = \int_{[0,\infty)} A_f \, d\lambda$$

Proof:

$\tilde{\mathcal{M}}$ is the σ-algebra of measurable subsets of $X \times \mathbb{R}$ corresponding to

$$\mu \times \lambda .$$

Set:

$$W = \{(x,t) : 0 < t < f(x)\} \subset X \times \mathbb{R} .$$

Need to show that W is measurable.

f measurable, $f \geq 0 \Rightarrow \exists \{f_k\}$ simple functions s.t.

$$f_k \leq f_{k+1} , \quad \lim_{k \to \infty} f_k = f$$

pointwise on X.

$$f_k = \sum_{j=1}^{n_k} a_{j,k} \chi_{E_{j,k}}$$

$E_{j,k}$ disjoint and measurable.

$$\Rightarrow W_k = \{(x,t) : 0 < t < f_k(x)\} = \bigcup_{j=1}^{n_k} E_{j,k} \times (0, a_{j,k})$$

$(x,t) \in W_k \Rightarrow 0 < t < f_k(x)$ & $x \in E_{j,k}$, some j

$\Rightarrow f_k(x) = a_{j,k} \Rightarrow t < a_{j,k} \Rightarrow (x,t) \in E_{j,k} \times (0, a_{j,k})$

$\Rightarrow W_K$ measurable in $X \times \mathbb{R}$, $\forall K$.

<u>Note:</u> $\chi_W = \lim_{K \to \infty} \chi_{W_K}$.

$W = \{(x,t) : 0 < t < f(x)\}$

$W_K = \{(x,t) : 0 < t < f_K(t)\}$.

Let $(x,t) \in X$.

Consider $f(x)$

- If $f(x) = 0$ then

  $f_K(x) = 0 \quad \forall K$

  $\Rightarrow x \notin W_K \quad \forall K$

  $\Rightarrow \chi_{W_K}(x,t) = 0 \quad \forall K$

  $\Rightarrow \chi_{W_K}(x,t) = 0 \to \chi_W(x,t) = 0$

- If $f(x) > 0 \Rightarrow \exists t$ s.t

  $0 < t < f(x)$ and $\chi_W(x,t) = 1$.

  Since $f_K(x) \uparrow f(x)$ then $\exists N, \delta > 0$

  $0 < t < \delta \leq f_K(x) \leq f(x) \quad \forall K \geq N$

  $0 < t < f_K(x), \quad \forall K \geq N$

  $(x,t) \in \chi_{W_K}, \quad \forall K \geq N$

  Thus, for $K \geq N$, $\chi_{W_K}(x,t) = 1 \to \chi_W(x,t)$

$\therefore W \in \tilde{m}$,

since χ_W is measurable.

$$\Updownarrow$$

$$W \in \tilde{m}$$

and χ_W is the limit of measurable functions, then χ_W is measurable.

Then:

$$\int_{[0,\infty)} \mu\{x : f(x) > t) \, d\lambda(t) =$$

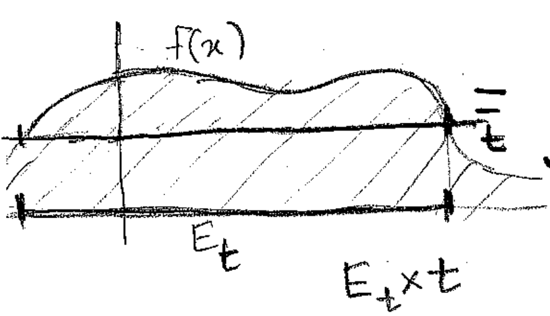
$$= \int_{[0,\infty)} \mu(E_t) \, d\lambda(t)$$

$t_1 < t_2$

$\{x : f(x) > t_2\} \subset \{x : f(x) > t_1\}$

$\Rightarrow \mu(E_{t_2}) \leq \mu(E_{t_1})$

$\therefore t \longmapsto \mu(E_t)$ is decreasing and non-negative.



$$= \int_{[0,\infty)} \left[\int_X \chi_W(x,t) \, d\mu \right] d\lambda(t)$$

For fixed t, this is a measurable function in X.

$$= \int_{\mathbb{R}} \int_X \chi_{W}(x,t)\, d\mu(x)\, d\lambda(t)$$

& $\chi_W \equiv 0$ for $t \le 0$.

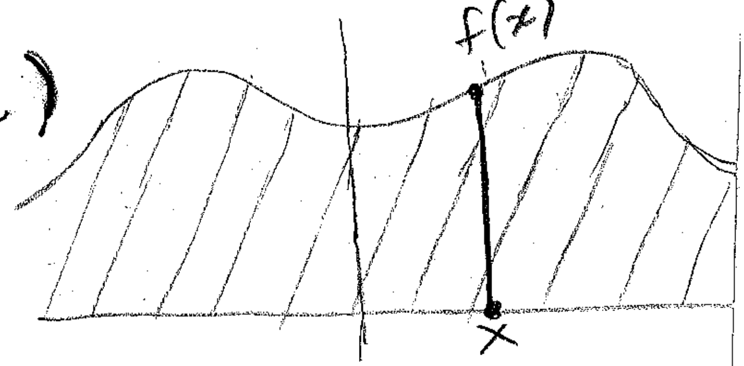
Fubini

$$= \int_X \int_{\mathbb{R}} \chi_W(x,t)\, d\lambda(t)\, d\mu(x)$$

fixed x

$$= \int_X \underbrace{\lambda(\{t : 0 < t < f(x)\})}_{= f(x)}\, d\mu(x)$$

$$= \int_X f(x)\, d\mu(x)$$



We have proved that:

$$\int_X f\, d\mu = \int_{[0,\infty)} \mu(\{x : f(x) > t\})\, d\lambda(t)$$

$$= \int_0^\infty \mu(\{x : f(x) > t\})\, d\lambda(t)$$

<u>Note:</u> Both sides could be ∞.

Remark: Let $\mu(X)<\infty$

$\Rightarrow \mu(\{x: f(x)>t\})<\infty, \quad \forall t$

$\Rightarrow \quad t \mapsto \mu(E_t)$ is monotone and bounded

$\Rightarrow t \mapsto \mu(E_t)$ is Riemann integrable on any compact interval in $[0,\infty)$.

In this case, using Theorem 159.1:

$$\int_X f\,d\mu = \int_0^\infty \mu(\{x: f(x)>t\})\,d\lambda(t)$$

$$= \lim_{b\to\infty} (R)\int_0^b \mu(E_t)\,dt$$

That is,

$\int_0^\infty \mu(E_t)\,d\lambda(t)$ can be computed as an improper Riemann integral.

Thm: f measurable

$1 \le p < \infty$.

Then:

$$\int_X |f|^p \, d\mu = p \int_{[0,\infty)} t^{p-1} \mu(\{x : |f(x)| > t\}) \, d\lambda(t)$$

Proof: Let

$W = \{(x,t) : 0 < t < |f(x)|\}$.

$(x,t) \longmapsto p t^{p-1} \chi_W(x,t)$

is $\mathcal{M}$-measurable.

$$\int_X |f|^p \, d\mu = \int_X |f|^p(x) \quad d\mu$$

$$= \int_X \left[\int_0^{|f(x)|} \frac{d}{dt} t^p \; dt \right] d\mu.$$

We use the Fundamental Theorem of Calculus for Riemann integrals:

$$\int_0^{|f(x)|} p\, t^{p-1} dt = \int_0^{|f(x)|} \frac{d}{dt} t^p dt$$

$$= |f(x)|^p$$

Hence:

$$\int_X |f|^p \, d\mu = \int_X \int_{(0,|f(x)|)} p\, t^{p-1} \, d\lambda(t)\, d\mu(x)$$

$$= \int_X \int_{\mathbb{R}} p\, t^{p-1} \chi_W(x,t)\, d\lambda(t)\, d\mu(x)$$

$$= \int_{\mathbb{R}} \int_X p\, t^{p-1} \chi_W(x,t)\, d\mu(x)\, d\lambda(t)$$

$$= p \int_{[0,\infty)} t^{p-1} \mu(\{x : |f(x)| > t\})\, d\lambda(t).$$