Lesson 32

Vitali's Covering Theorem.

Notation: If B is a closed ball in $\mathbb{R}^n$, we write $\hat{B}$ to denote the concentric closed ball with radius 5 times the radius of B.

Thm: Vitali's Covering Theorem.
Let $\mathcal{F}$ be a collection of non-degenerate closed balls in $\mathbb{R}^n$ with

$$\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < \infty,$$

Then there exists a countable family $\mathcal{G}$ of disjoints balls in $\mathcal{F}$ such that

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} \hat{B}$$

Proof: Let

$$D := \sup\{\operatorname{diam} B \mid B \in \mathcal{F}\}$$

Define:

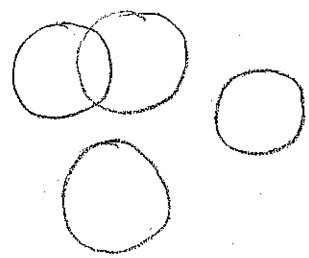
$$\mathcal{F}_j = \left\{B \in \mathcal{F} \;\middle|\; \frac{D}{2^j} < \operatorname{diam} B \le \frac{D}{2^{j-1}}\right\},$$

$j = 1, 2, 3, \ldots$

We define $\mathcal{G}_j \subset \mathcal{F}_j$ as follows:

(a) Let $\mathcal{G}_1$ be any maximal disjoint collection of balls in $\mathcal{F}_1$

$\mathcal{G}_1$ is maximal in the sense that it is not a subset of any other disjoint collection of balls in $\mathcal{F}_1$. The existence of this maximal collection requires the axiom of choice (see Chapter 1).



(b) Assuming that $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{k-1}$ have been selected, we choose $\mathcal{G}_k$ to be any maximal disjoint subcollection of

$$\{ B \in \mathcal{F}_k \mid B \cap B' = \emptyset, \ \forall B' \in \bigcup_{j=1}^{k-1} \mathcal{G}_j \} \quad (*)$$

Define now:

$$\mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{G}_j$$

$\mathcal{G}$ is a collection of disjoint balls and $\mathcal{G} \subset \mathcal{F}$.

<u>Claim</u>: For each $B \in \mathcal{F}$, $\exists B' \in \mathcal{G}$ so that $B \cap B' \neq \emptyset$ and

$$B \subset \hat{B'}$$

Clearly, this claim implies the desired Theorem. We now proceed to proof this claim:

Fix $B \in \mathcal{F}$

Then $B \in \tilde{\mathcal{F}}_j$, ~~for~~ some j.

If $B \in \mathcal{G}_j$ the the claim is obvious. If $B \notin \mathcal{G}_j$, then by the maximality of $\mathcal{G}_j$, $\exists B' \in \bigcup_{k=1}^{j} \mathcal{G}_k$ with

$$B \cap B' \neq \emptyset.$$

(For otherwise B' would belong to $\mathcal{G}_j$, see $(*)$).

Let us say that $B' \in \mathcal{G}_{k_0}$, with $k_0 \in \{1, 2, \ldots, j\}$. Then $k_0 \leq j$ and:

$$\frac{D}{2^j} \leq \frac{D}{2^{k_0}} < \operatorname{diam} B' \leq \frac{D}{2^{k_0 - 1}},$$

$$\frac{D}{2^j} < \operatorname{diam} B \leq \frac{D}{2^{j-1}}$$

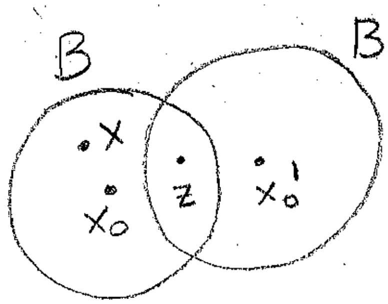
Hence:

$$\operatorname{diam} B \le \frac{D}{2^{j-1}} = \frac{2D}{2^j} < 2 \operatorname{diam} B'$$

$\Rightarrow$

$$\boxed{\operatorname{diam} B < 2 \operatorname{diam} B'} \qquad (2)$$

We now prove that $\underline{B \subset \hat{B}'}$. Let x_0, x_0' be the centers of B and B' respectively:



Let $x \in B$. Then

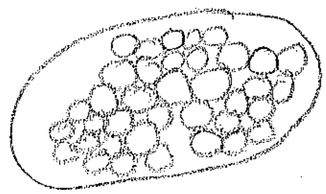
$$|x - x_0'| \le |x - x_0| + |x_0 - z| + |z - x_0'|$$
$$\le \frac{\operatorname{diam} B}{2} + \frac{\operatorname{diam} B}{2} + \frac{\operatorname{diam} B'}{2}$$
$$\le \operatorname{diam} B' + \operatorname{diam} B' + \frac{\operatorname{diam} B'}{2}; \text{ by } (2)$$
$$= \frac{5}{2} \operatorname{diam} B' = 5 (\text{radius } B')$$
$$= \text{radius } (\hat{B}')$$

$\therefore \; x \in \hat{B}'$

$\therefore \; \underline{B \subset \hat{B}'}$ ∎

<u>Corollary</u>: Let $U \subset \mathbb{R}^n$ be open, $\delta > 0$. There exists a countable collection $\mathcal{G}$ of disjoint closed balls in U such that $\operatorname{diam} B \leq \delta$ $\forall B \in \mathcal{G}$ and

$$\lambda_n \Big(U \setminus \bigcup_{B \in \mathcal{G}} B \Big) = 0$$



Proof

Fix $1 - \frac{1}{5^n} < \theta < 1$. Assume first $\lambda(U) < \infty$.

<u>Claim</u>: $\exists$ a finite collection $\{B_i\}_{i=1}^{M_1}$ of disjoint closed balls in U such that $\operatorname{diam}(B_i) < \delta$ $\forall i = 1, \ldots, M_1$ and

$$\lambda \Big(U \setminus \bigcup_{i=1}^{M_1} B_i \Big) \leq \theta \lambda(U)$$

<u>Proof of Claim</u>: Let

$$\mathcal{F}_1 = \{ B \mid B \subset U, \operatorname{diam} B < \delta \}$$

By Vitali's Covering Theorem, $\exists$ a countable disjoint family

$\mathcal{G}_1 \subset \mathcal{F}_1$ such that

$$U \subset \bigcup_{B \in \mathcal{G}_1} \hat{B}.$$

Therefore,

$$\lambda(U) \leq \sum_{B \in \mathcal{G}_1} \lambda(\hat{B}) = 5^n \sum_{B \in \mathcal{G}_1} \lambda(B) = 5^n \lambda\Big(\bigcup_{B \in \mathcal{G}_1} B\Big)$$

Thus

$$\lambda\Big(\bigcup_{B \in \mathcal{G}_1} B\Big) \geq \frac{1}{5^n}\lambda(U)$$

$$\therefore \quad \lambda\Big(U \setminus \bigcup_{B \in \mathcal{G}_1} B\Big) \leq \Big(1 - \frac{1}{5^n}\Big)\lambda(U)$$

Since $\mathcal{G}_1$ is countable and $\sum_{B \in \mathcal{G}_1} \lambda(B) < \infty$, then $\exists$ $B_1, B_2, \ldots, B_{M_1}$ in $\mathcal{G}_1$ such that

$$\lambda\Big(U \setminus \bigcup_{i=1}^{M_1} B_i\Big) \leq \theta\,\lambda(U),$$ which proves the claim.

Define now:

$$U_2 := U \setminus \bigcup_{i=1}^{M_1} B_i$$

$$\mathcal{F}_2 := \{B \mid B \subset U_2,\ \operatorname{diam} B < \delta\},$$

and proceed as above to find disjoint balls $B_{M_1+1}, \ldots, B_{M_2}$ in $\mathcal{F}_2$ such that:

$$\lambda\Big(U \setminus \bigcup_{i=1}^{M_2} B_i\Big) = \lambda\Big(U_2 \setminus \bigcup_{i=M_1+1}^{M_2} B_i\Big) \leq \theta\,\lambda(U_2) \leq \theta^2 \lambda(U).$$

Continue this process to obtain a countable collection of disjoints balls $\mathcal{G} = \{ B_1, \ldots, B_{M_1}, B_{M_1+1}, \ldots, B_{M_2}, B_{M_2+1}, \ldots, B_{M_3}, \ldots$
$\ldots, B_{M_4}, \ldots, B_{M_K}, \ldots \}$ such that:

$$\lambda\left(U \setminus \bigcup_{i=1}^{M_K} B_i\right) \leq \theta^K \lambda(U) \quad \forall K$$

Letting $K \to \infty$ we obtain

$$\lambda\left(U \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0 .$$

If $\lambda(U) = \infty$ then we apply the above reasoning to the sets,

$$U_m = \{ x \in U : m < |x| < m+1 \},$$

$m = 0, 1, 2, \ldots$