7.2 Lebesgue points.

Def: Let $f \in L^1(\mathbb{R}^n)$. We Define Mf, the maximal function:

$$Mf(x) := \sup_{r>0} ⨍_{B(x,r)} |f| \, d\lambda$$

where $$⨍_E |f| \, d\lambda = \frac{1}{\lambda(E)} \int_E |f| \, d\lambda$$

$Mf : \mathbb{R}^n \to \overline{\mathbb{R}}$ non-negative function.

- Mf is measurable
- $Mf \notin L^1(\mathbb{R}^n)$

Thm: (Hardy- Littlewood).

Let $f \in L^1(\mathbb{R}^n)$. Then

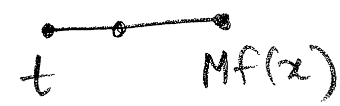
$$\lambda[\{Mf > t\}] \leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f| \, d\lambda \quad \forall t > 0.$$

Proof: Fix $t>0$

Let $x \in \{Mf > t\}$

$\therefore \quad Mf(x) > t$

$\therefore \quad \sup_{r>0} ⨍_{B(x,r)} |f| \, d\lambda > t$



$\Rightarrow \exists$ a ball B_x centered at x s.t.

$$⨍_{B_x} |f| \, d\lambda > t$$

$$\therefore \quad \frac{1}{\lambda(B_x)} \int_{B_x} |f| \, d\lambda > t$$

$$\therefore \quad \frac{1}{t} \int_{B_x} |f| \, d\lambda > \lambda(B_x)$$

Let $\quad \mathcal{F} = \{B_x\}$

Note: $f \in L^1(\mathbb{R}^n) \Rightarrow$
$\sup \{\operatorname{diam} B_x \mid B_x \in \mathcal{F}\} < \infty$

Then, from Vitali's Covering Theorem it follows that there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ s.t.

$$\bigcup_{B_x \in \mathcal{F}} B_x \subset \bigcup_{B_{x_i} \in \mathcal{G}} \hat{B}_{x_i}$$

Since $\{Mf > t\} \subset \bigcup_{B_x \in \mathcal{F}} B_x$, then

$$\{Mf > t\} \subset \bigcup_{B_{x_i} \in \mathcal{G}} \hat{B}_{x_i}$$

$$\Rightarrow \lambda\{Mf > t\} \leq \lambda\left(\bigcup \hat{B}_{x_i}\right)$$

$$\leq \sum_{i=1}^{\infty} \lambda(\hat{B}_{x_i})$$

$$= 5^n \sum_{i=1}^{\infty} \lambda(B_{x_i})$$

$$< \frac{5^n}{t} \sum_{i=1}^{\infty} \int_{B_{x_i}} |f| \, d\lambda$$

$$\leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f| \, d\lambda \quad \blacksquare$$

If $f \in L^1_{loc}(\mathbb{R}^n)$ is continuous,

then
$$\lim_{r \to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(y)\, d\lambda(y) = f(x)$$

f continuous at $x \in \mathbb{R}^n$. Let $\varepsilon > 0$;

then $\exists\, \delta > 0$ s.t $y \in B(x, \delta)$

implies
$$|f(y) - f(x)| < \varepsilon$$

$$\left| \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} (f(y) - f(x))\, dy \right|$$

$$\leq \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy$$

$$\leq \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} \varepsilon\, dy, \quad \text{for } r < \delta$$

$$= \varepsilon \frac{\lambda(B(x,r))}{\lambda(B(x,r))} = \varepsilon$$

$$\therefore \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} (f(y) - f(x))\, dy \to 0$$

$$\therefore \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(y) \longrightarrow f(x).$$

Thm: If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r \to 0} \fint_{B(x,r)} f(y)\, d\lambda(y) = f(x)$$

for a.e. $x \in \mathbb{R}^n$.

<u>Proof</u>: Let $\varepsilon > 0$.

Since continuous functions are dense in $L^1(\mathbb{R}^n)$ we have that:

$\exists$ continuous function $g \in L^1(\mathbb{R}^n)$ s.t.:

$$\int_{\mathbb{R}^n} |f(y) - g(y)|\, d\lambda(y) < \varepsilon.$$

Since g is continuous:

$$\lim_{r \to 0} \fint_{B(x,r)} g(y)\, d\lambda(y) = g(x) \quad \forall x,$$

$$\limsup_{r \to 0} \left| \fint_{B(x,r)} f(y)\, d\lambda(y) - f(x) \right|$$

$$= \limsup_{r \to 0} \left| \fint_{B(x,r)} f(y)\, d\lambda \quad - f(x) \right|$$

$$= \limsup_{r \to 0} \left| \fint_{B(x,r)} [f(y) - g(y)]\, d\lambda(y) \right.$$

$$\left. + \left(\fint_{B(x,r)} g(y)\, d\lambda(y) - g(x) \right) + \quad g(x) - f(x) \right|$$

$$\leq \limsup_{r \to 0} \fint_{B(x,r)} |f(y) - g(y)|\, d\lambda$$

$$+ \limsup_{r \to 0} \left| \fint_{B(x,r)} g(y)\, d\lambda \quad - g(x) \right|$$

$$+ \limsup_{r \to 0} |g(x) - f(x)|$$

$$\leq \quad M(f-g)(x) + 0 + |f(x) - g(x)|$$

For each $t > 0$, let

$$E_t = \{x : \limsup_{r \to 0} \left| \fint_{B(x,r)} f(y)\, d\lambda - f(x) \right| > t\}$$

$$F_t = \{x : |f(x) - g(x)| > t\}$$

$$H_t = \{x : M(f-g)(x) > t\}.$$

$$E_t \subset F_{t/2} \cup H_{t/2}$$

$$x \in E_t \implies M(f-g)(x) + |f(x)-g(x)| > t$$

$$\implies M(f-g)(x) > \frac{t}{2} \quad \text{or}$$

$$|f(x) - g(x)| > \frac{t}{2}$$

$$= x \in F_{t/2} \quad \text{or} \quad x \in H_{t/2}$$

$$= x \in F_{t/2} \cup H_{t/2}$$

We have:

$$\lambda(H_{t/2}) = \lambda\left[\{M(f-g) > \frac{t}{2}\right] \le \frac{2}{t} 5^n \int_{\mathbb{R}^n} |f-g|\, d\lambda$$

$$\le \frac{2 5^n \varepsilon}{t}$$

$$\lambda(F_{t/2}) = \lambda\{x : |f(x) - g(x)| > \tfrac{t}{2}\}$$

$$\frac{t}{2}\lambda(F_{t/2}) \le \int_{F_{t/2}} |f(y) - g(y)|\, d\lambda$$

$$\le \int_{\mathbb{R}^n} |f-g|\, d\lambda$$

$$< \varepsilon$$

$$\therefore \lambda(F_{t/2}) < \frac{2\varepsilon}{t}$$

$\Rightarrow$

$$\lambda(E_t) \le \lambda(F_{t/2}) + \lambda(H_{t/2})$$

$$\le \frac{2 \cdot 5^n \varepsilon}{t} + \frac{2\varepsilon}{t}$$

ε arbitrary $\Rightarrow$ $\lambda(E_t) = 0$.

We conclude

$$\lambda(E_t) = 0 \quad \forall t > 0$$

$$\therefore \limsup_{r \to 0} \left| ⨍_{B(x,r)} f(y)\, d\lambda(y) - f(x) \right| = 0 \quad a.e.\ x$$

$$\therefore \liminf_{r \to 0} \left| \fint_{B(x,r)} f(y)\, d\lambda(y) - f(x) \right| = 0 \quad \text{a.e. } x$$

$$\therefore \lim_{r \to 0} \left| \fint_{B(x,r)} f(y)\, d\lambda(y) - f(x) \right| = 0 \quad \text{a.e. } x$$

$$\Rightarrow \lim_{r \to 0} \fint_{B(x,r)} f(y)\, d\lambda(y) = f(x) \quad \text{a.e. } x$$

<u>Thm</u>: If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)|\, d\lambda(y) = 0$$

for a.e. $x \in \mathbb{R}^n$

<u>Proof</u>: For every $\rho \in \mathbb{Q}$, $\exists E_\rho$, $\lambda(E_\rho) = 0$ s.t

$$\Rightarrow \lim_{r \to 0} \fint_{B(x,r)} |f(y) - \rho|\, d\lambda = |f(x) - \rho| \quad \forall x \in E_\rho^c$$

Let :

$$E := \bigcup_{\rho \in \mathbb{Q}} E_\rho$$

$$\lambda(E) = 0 .$$

Take $x \notin E$.

$$\limsup_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)| \, d\lambda(y)$$

$$\leq \limsup_{r \to 0} \fint_{B(x,r)} |f(y) - \rho| \, d\lambda(y)$$

$$+ \limsup_{r \to 0} \fint_{B(x,r)} |\rho - f(x)| \, d\lambda(y)$$

$$\leq 2 |f(x) - \rho| \quad \forall \rho \in \mathbb{Q}$$

Since $\exists \, \rho_k$, such that $\rho_k \to f(x)$

we conclude :

$$\limsup_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)| \, d\lambda(y) = 0$$