Lesson 34

Radon Nikodym Derivative.

We consider in this section only measures in $\mathbb{R}^n$.

<u>Definition</u>: A Borel measure on $\mathbb{R}^n$ that is finite on compact sets is called a Radon measure

<u>Ex</u>: Lebesgue measure λ is a Radon measure but H^s, $0<s<n$ is not, since for example:

$$H^2(Q)=\infty, \quad Q=[0,1]\times[0,1]\times[0,1]\subset\mathbb{R}^3$$

Let μ be a Radon measure in $\mathbb{R}^n$ such that

$$\mu \ll \lambda$$

The Radon-Nikodym Theorem implies that there exists f measurable such that either f^+ or f^- is integrable and:

$$(*) \quad \mu(E)=\int_E f(y)\,d\lambda(y), \quad E \text{ Borel measurable}$$

Notation: In (*), the function f is denoted as

$$D_\lambda \mu \quad \text{or} \quad \frac{d\mu}{d\lambda}$$

We have the following:

Note that since μ is a Radon measure then $f \in L^1_{loc}(\mathbb{R}^n)$. In fact, for any compact set $K \subset \mathbb{R}^n$,

$$\mu(K) = \int_K f(y)\, d\lambda(y) < \infty.$$

We have the following:

Lemma: If $\mu << \lambda$, μ Radon measure. Then,

$$f(x) = D_\lambda \mu(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}, \quad \lambda\text{-a.e. } x.$$

Proof: Since $f \in L^1_{loc}(\mathbb{R}^n)$ then from the Theorem proved last class:

$$\lim_{r \to 0} ⨍_{B(x,r)} f(y)\, d\lambda(y) = f(x), \quad \lambda\text{-a.e. } x \in \mathbb{R}^n$$

Since:

$$\mu(B(x,r)) = \int_{B(x,r)} f(y)\, d\lambda(y)$$

then,

$$\frac{\mu(B(x,r))}{\lambda(B(x,r))} = \frac{\int_{B(x,r)} f(y)\, d\lambda(y)}{\lambda(B(x,r))}$$

$\therefore$

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} = \lim_{r\to 0} ⨍ f(y)\, d\lambda(y) = f(x),$$

for λ-a.e. x. ∎

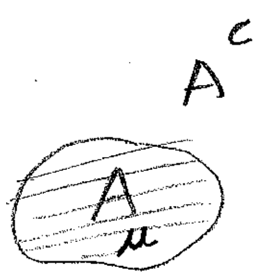
<u>Thm</u>: Let μ be a Radon measure that is singular with respect to λ. That is, μ is concentrated on a Borel set A, and

$$\mu(A^c) = \lambda(A) = 0,$$

then:

$$D_\lambda \mu(x) := \lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} = 0, \quad \lambda\text{-a.e. } x.$$

Proof:



$\lambda(A)=0$

Define:

$$E_k = A^c \cap \left\{x;\ \limsup_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} > \frac{1}{k}\right\},\ k=1,2,\ldots$$

If we show that:

$$\lambda(E_k)=0 \quad \forall k,$$

then $\lambda\left(\bigcup_{k=1}^{\infty} E_k\right)=0$ and for $x\in A^c \setminus \bigcup_{k=1}^{\infty} E_k$

we have:

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}=0, \quad \text{and}$$

hence, since $\lambda(A)=0$ we can conclude:

$$D_\lambda \mu(x)=0 \quad \lambda\text{-a.e. } x.$$

The function:

$$h_r(x) = \frac{\mu(B(x,r))}{\lambda(B(x,r))}$$

is lower semicontinuous (see Ex. 4.11 + Section 3.8 on lower semicontinuous functions).

From Chapter 3:

For every $t \in \mathbb{R}$

$\{h_r > t\} \subset \mathbb{R}^n$ is open

$\therefore \{h_r > t\} \in \mathcal{B}$; since every open set is Borel set

$\therefore$ $h_r : \mathbb{R}^n \to \mathbb{R}$

is Borel measurable

$\therefore$ $h := \limsup_{r \to 0} h_r$

is Borel measurable

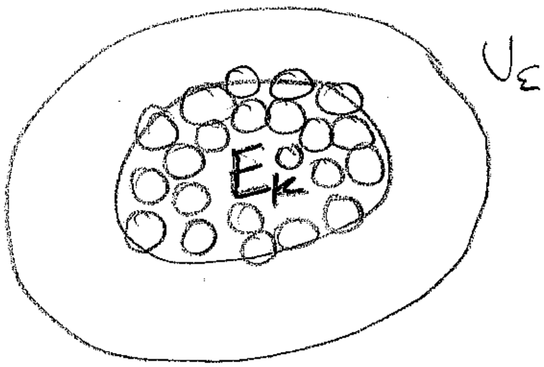
$\therefore \{x : h(x) > \frac{1}{k}\}$ is Borel

$\therefore$ <u>E_k is Borel</u>

Let $\varepsilon > 0$.

Refer now to Theorems 115.1, 108.1 and Lemma 1 (Page 8 from Measure Theory and Fine properties of functions) to deduce that, since μ is a Borel measure with the property that $\mathbb{R}^n = \bigcup_{n=1}^{\infty} B(0,n)$, $\mu(B(0,n)) < \infty$ then $\exists U_\varepsilon$, open set, $E_k \subset U_\varepsilon$ and $\mu(U_\varepsilon \setminus E_k) < \varepsilon$

$\therefore \quad \mu(U_\varepsilon \setminus E_k) = \mu(U_\varepsilon) \setminus \mu(E_k)$
$= \mu(U_\varepsilon) - 0 = \mu(U_\varepsilon) < \varepsilon.$



for each $x \in E_k$, $\exists B(x, r_x)$, $r_x < 1$ such that:

$$\begin{cases} B(x, r_x) \subset U_\varepsilon \\ \lambda(B(x, r_x)) < K \mu(B(x, r_x)) \end{cases}$$

So

$$E_k \subset \bigcup_x B(x, r_x)$$

Vitali's Covering Theorem $\Rightarrow$

$$E_k \subset \bigcup_{i=1}^{\infty} \hat{B}(x_i, r_{x_i}),$$

for some countable disjoint collection $\{B(x_i, r_{x_i})\}$

Thus:

$$\begin{aligned} \lambda(E_k) &\leq \sum_{i=1}^{\infty} \lambda(\hat{B}(x_i, r_{x_i})) \\ &= 5^n \sum_{i=1}^{\infty} \lambda(B(x_i, r_{x_i})) \\ &< 5^n K \sum_{i=1}^{\infty} \mu(B(x_i, r_{x_i})) \\ &\leq 5^n K \mu(U_\varepsilon) \leq 5^n K \varepsilon \Rightarrow \lambda(E_k) = 0 \ \blacksquare \end{aligned}$$