Lesson 36

Functions of Bounded Variation.

<u>Def</u>: Let $f: [a,b] \to \mathbb{R}$. The total variation of f from a to x, $x \le b$, is defined by:

$$V_f(a;x) = \sup \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})|$$

where the supremum is taken over all finite sequences $a = t_0 < t_1 < \cdots < t_k = x$

We say that f is a function of bounded variation on $[a,b]$; that is,

$$f \in BV([a,b])$$

if $\quad V_f(a;b) < \infty$.

<u>Remark</u>: If $f \in BV([a,b])$ then f is bounded.

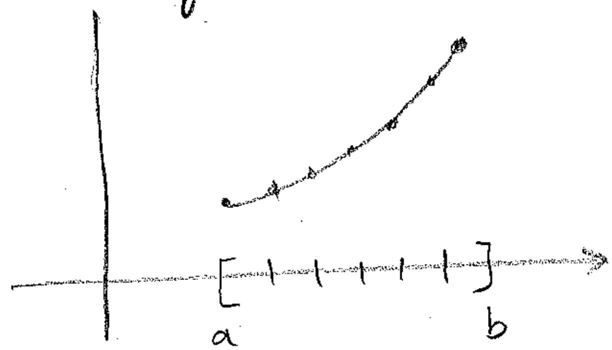
Indeed, let $x \in [a,b]$. Then

$$|f(x)| - |f(a)| \le |f(x) - f(a)| \le V_f(a;x) \le V(a;b)$$

$$\Rightarrow |f(x)| \le |f(a)| + V_f(a;b), \quad x \in [a,b]$$

$\Rightarrow$ f is bounded.

<u>Remark</u>: Let $f:[a,b] \to \mathbb{R}$ be non-decreasing. Then $f \in BV([a,b])$



Indeed, on the one hand:

$$f(b)-f(a) \leq V_f(a;b).$$

On the other hand, since f is non-decreasing we have:

$$\sum_{i=1}^{k} |f(t_i)-f(t_{i-1})| \leq f(b)-f(a),$$

for every partition $a_0 = t_0 < t_1 < \cdots < t_k = b$

Thus:

$$V_f(a;b) \leq f(b)-f(a)$$

$\therefore$ $V_f(a,b) = f(b)-f(a)$ if f is non-decreasing

<u>Remark</u>: If $f,g \in BV([a,b])$ then $f+g \in BV([a,b])$.

Thm: Let $f \in BV([a,b])$. Then

$$f = f_1 - f_2,$$

where both f_1 and f_2 are non-decreasing.

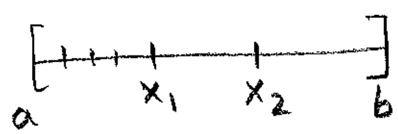
Proof:

Let $x_1 < x_2 \leq b$. We use the notation $V_f(x) := V_f(a;x)$.

Let $a = t_0 < t_1 < \cdots < t_k = x_1$ be any partition of $[a, x_1]$.

Clearly:

$$\sum_{i=1}^{k} |f(t_i) - f(t_{i-1})| + |f(x_2) - f(x_1)| \leq V_f(x_2)$$



$$\therefore \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})| \leq \underbrace{V_f(x_2) - |f(x_2) - f(x_1)|}_{\text{Upper bound}}$$

$$\therefore \quad V_f(x_1) \leq V_f(x_2) - |f(x_2) - f(x_1)|$$

In particular,

$$V_f(x_2) - V_f(x_1) \geq |f(x_2) - f(x_1)| \longrightarrow (1)$$

From (1) we have: (36.4)

$V_f(x_2) - V_f(x_1) \geq f(x_2) - f(x_1) \rightarrow (2)$

and

$V_f(x_2) - V_f(x_1) \geq f(x_1) - f(x_2) \longrightarrow (3)$

From (2) and (3):

$$\boxed{\begin{gathered} V_f(x_2) - f(x_2) \geq V_f(x_1) - f(x_1) \\ \text{and} \\ V_f(x_2) + f(x_2) \geq V_f(x_1) + f(x_1) \end{gathered}}$$

$\therefore$ The functions:

$V_f - f$ and $V_f + f$ are

non-decreasing.

Define:

$$f_1 := \tfrac{1}{2}(V_f + f), \quad f_2 := \tfrac{1}{2}(V_f - f)$$

Therefore:

$$\boxed{f = f_1 - f_2}$$

and f_1, f_2 are non-decreasing.

Recall the second part of the Fundamental Theorem of Calculus for Riemann integrals:

If $f \in \mathcal{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that

$$F' = f$$

then:

$$(R)\int_a^b f(x)\,dx = F(b) - F(a)$$

We need to prove an analogous Theorem for Lebesgue integrals.

<u>Definition</u>: A function $f: [a,b] \to \mathbb{R}$ is said to be absolutely continuous on $[a,b]$, written as:

$$f \in AC([a,b])$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| < \varepsilon$$

for any collection of non-overlapping intervals $[a_1,b_1], [a_2,b_2], \ldots, [a_k,b_k]$ on $[a,b]$ with $\sum_{i=1}^{k} |b_i - a_i| < \delta$.

Recall also the definition:

Let $f: X \to Y$, X, Y metric spaces.
We say that f is uniformly continuous on X if:
$\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon.$$
$$p, q \in X$$

<u>Remark</u>: If f is absolutely continuous then f is uniformly continuous.

However, the converse is not true. For example, consider the Cantor-Lebesgue function:

$$f: [0,1] \to [0,1]$$

Recall that $f(C) = [0,1]$, C is the cantor set. Since f is continuous on the compact set $[0,1]$ then f is uniformly continuous on $[0,1]$. But $f \notin AC([0,1])$ because:

$$\lambda(C) = 0 \text{ but } \lambda(f(C)) \neq 0.$$

Indeed, see the next theorem.

Def: A function $f: [a,b] \to \mathbb{R}$ is said to satisfy <u>condition N</u> if

$$E \subset [a,b],\ \lambda(E)=0 \implies \lambda(f(E))=0$$

Ex: If f is Lipschitz then f satisfies Condition N.

Thm: If $f \in AC([a,b]) \Rightarrow f$ satisfies condition N.

Proof: Choose $\varepsilon > 0$ and let $\delta > 0$ given by the definition of absolute continuity.

Let $E \subset [a,b]$, $\lambda(E)=0$

Let U be an open set, $E \subset U$ such that:

$$\lambda(U \setminus E) = \lambda(U) - \lambda(E) < \delta$$

We have:

$$U = \bigcup_{i=1}^{\infty} (a_i, b_i),$$ disjoint union of open intervals.

Since f is continuous, then f attains a max and a min at t^i_{max}, t^i_{min} on $[a_i, b_i]$.

Thus, $[a_i, b_i]$ contains an interval:

$$[s_{i,1}, s_{i,2}], \quad s_{i,1} = t^i_{min}, \quad s_{i,2} = t^i_{max}$$

(or viceversa since $t^i_{min} \leq t^i_{max}$ or $t^i_{max} \leq t^i_{min}$).

Hence:

$$f([a_i, b_i]) = [f(s_{i,1}), f(s_{i,2})]$$

Thus:

$$\lambda(f(E)) \leq \sum_{i=1}^{\infty} \lambda(f([a_i, b_i])), \quad E \subset \bigcup_{i=1}^{\infty} [a_i, b_i]$$

$$= \sum_{i=1}^{\infty} \lambda([f(s_{i,1}), f(s_{i,2})])$$

$$< \varepsilon, \quad \text{since} \quad \sum_{i=1}^{\infty} |s_{i,2} - s_{i,1}| < \delta$$

Since ε is arbitrary we conclude

$$\lambda(f(E)) = 0.$$