Lesson 38

<u>Theorem</u>: Let $f \in BV([a,b])$.

Then

$$\int_a^b |f'(x)|\, d\lambda(t) \le V_f(b)$$

<u>Proof</u>:

Consider first the case when

f is non-decreasing.

We extend f by defining $f(x) = f(b)$, $x > b$.

Define:

$$g_i(x) = \frac{f(x+\frac{1}{i}) - f(x)}{\frac{1}{i}}, \quad i = 1, 2, \ldots \qquad (1)$$

Since f is non-decreasing then:

f is continuous except possibly on a countable set.

Recall the following:

<u>Lemma</u>: If $g: \mathbb{R}^1 \to \mathbb{R}$ is continuous except on a set of measure zero then g is Lebesgue measurable.

Indeed, to see that the Lemma is true

we write $\mathbb{R}^n = C \cup D$, where $\lambda(D)=0$.

Since g restricted to C is continuos in the relative topology, we define:

$$\tilde{g} = g|_C .$$

Then, for every open interval (c,d):
$\tilde{g}^{-1}(c,d)$ is open in C, and thus:

$$\tilde{g}^{-1}(c,d) = U \cap C, \quad U \subset \mathbb{R}^n \text{ open set.}$$

Note that:

$$g^{-1}(c,d) = [g^{-1}(c,d) \cap C] \cup [g^{-1}(c,d) \cap D]$$
$$= \tilde{g}^{-1}(c,d) \cup [g^{-1}(c,d) \cap D]$$
$$= [U \cap C] \cup [g^{-1}(c,d) \cap D]$$

Note that $U \cap C$ is Lebesgue measurable and $g^{-1}(c,d) \cap D$ is also Lebesgue measurable because it has measure zero. Hence we conclude that f is Lebesgue measurable. ∎

Going back to our <u>non-decreasing f</u>, since f is continuous almost every-where then, by Lemma, f is Lebesgue measurable. Actually, f is Borel measurable since in this case the set D in the Lemma is countable (and hence Borel) and thus $f^{-1}(a,b)$ is Borel.

Since f is Borel, then g_i is Borel and hence:

$$u(x) := \limsup_{i\to\infty} g_i(x) \text{ is Borel measurable}$$

and

$$v(x) := \liminf_{i\to\infty} g_i(x) \text{ is Borel measurable.}$$

Since f is non-decreasing we proved earlier that:

f is differentiable <u>λ-almost everywhere.</u>

Hence

$$f'(x) = u(x) = v(x), \quad \lambda\text{-a.e. } x$$

Thus, u is Borel measurable and hence Lebesgue measurable. Since $f' = u$ λ-a.e, we conclude that f' is <u>Lebesgue</u> measurable (but we can not conclude that f' is Borel measurable).

In the general case, if $f \in BV([a,b])$ then we write:

$$f = f_1 - f_2, \quad f_1, f_2 \text{ non-decreasing}$$

and from the above discussion we conclude:

(a) f is Borel measurable

(b) f' is Lebesgue measurable since:

$$f'(x) = f_1'(x) - f_2'(x), \quad \lambda\text{-a.e. } x$$

and f_1', f_2' are Lebesgue measurable

We now proceed to estimate:

$$\int_a^b |f'(x)| \, d\lambda(x).$$

We consider again first the case when f is non-decreasing. Then, using (1):

$$\int_a^b \liminf g_i \, d\lambda(x) \leq \liminf_{i\to\infty} \int_a^b g_i(x) \, d\lambda(x),$$ using Fatou's since $g_i \geq 0$

$$\| \quad \int_a^b f'(x) \, d\lambda(x)$$

$$\therefore \int_a^b f'(x) \, d\lambda(x) \leq \liminf_{i\to\infty} i \int_a^b \left[f(x+\tfrac{1}{i}) - f(x) \right] d\lambda(x)$$

$$= \liminf_{i\to\infty} i \left[\int_{a+1/i}^{b+1/i} f(x) \, d\lambda(x) - \int_a^b f(x) \, d\lambda(x) \right];$$ By a change of variables for Riemann integrals.

$$= \liminf_{i\to\infty} i \left[\int_b^{b+1/i} f(x) \, d\lambda(x) - \int_a^{a+1/i} f(x) \, d\lambda(x) \right]$$

$\therefore$

$$\int_a^b f'(x)\,d\lambda(x) \le \liminf_{i\to\infty} i\left[\int_b^{b+1/i} f(b)\,d\lambda(x) - \int_a^{a+1/i} f(a)\,d\lambda(x)\right]$$

$$= \liminf_{i\to\infty} i\left[\frac{f(b)}{i} - \frac{f(a)}{i}\right]$$

$$= f(b)-f(a)$$

Hence

(2) $\boxed{\int_a^b f'(x)\,d\lambda(x) \le f(b)-f(a)}$, f non-decreasing.

In the general case, take $f \in BV([a,b])$.

Recall that:

$$V_f(x) = \sup \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})|, \text{ where}$$

the sup is taken over all partitions $a = t_0 < t_1 < \cdots < t_k = x$.

Since $x \mapsto V_f(x)$ is non-decreasing then:

$V_f'(x)$ exists for λ-a.e. x.

Recall that:

$$f = f_1 - f_2,$$

where

(3) $\boxed{f_1 = \frac{1}{2}(V_f + f), \quad f_2 = \frac{1}{2}(V_f - f)}$

<u>Claim</u>: $|f'| = V_f'$ λ-a.e.

We first show that $|f'(x)| \leq V_f'(x)$ for λ-a.e. x. Indeed we compute, for λ-a.e. x:

$$|f'(x)| = |f_1'(x) - f_2'(x)| \leq |f_1'(x)| + |f_2'(x)|$$

$$= f_1'(x) + f_2'(x)$$ since f_1, f_2 non-decreasing $\Rightarrow$ $f_1', f_2' \geq 0$

$$= \frac{1}{2}(V_f'(x) + f'(x)) + \frac{1}{2}(V_f'(x) - f'(x)) \text{ ; by (3)}$$

$$= V_f'(x).$$

$$\therefore \quad |f'(x)| \leq V_f'(x) \quad , \lambda\text{-a.e. } x. \longrightarrow (4)$$

To prove the other inequality we consider the set:

$$E = [a,b] \cap \{t : V_f'(x) > |f'(x)|\}$$

We refer to the book for the proof that:
$\lambda(E) = 0$.

Hence:

$$V_f'(x) \leq |f'(x)| \text{ for } \lambda\text{-a.e. } x \longrightarrow (5)$$

$$(4) + (5) \Rightarrow \boxed{V_f'(x) = |f'(x)| \quad , \lambda\text{-a.e. } x.}$$

Finally, from (2) and previous claim we compute:

$$\int_a^b |f'(x)|\, d\lambda(x) = \int_a^b V_f'(x)\, d\lambda(x) \quad , \text{ by } \underline{\text{Claim}}$$

$$= \int_a^b [f_1'(x) + f_2'(x)]\, d\lambda(x) \quad ; \text{ by } (3)$$

$$= \int_a^b f_1'(x)\, d\lambda(x) + \int_a^b f_2'(x)\, d\lambda(x)$$

$$\leq f_1(b) - f_1(a) + f_2(b) - f_2(a) \quad ; \text{ by } (2)$$

$$= [f_1(b) + f_2(b)] - [f_1(a) + f_2(a)]$$

$$= V_f(b) - V_f(a) \quad ; \text{ Since } V_f = f_1 + f_2 \text{ by } (3).$$

$$= V_f(b) - 0 \quad ; \text{ since } V_f(a;a) = 0$$

Hence, we conclude the desired estimate:

$$\int_a^b |f'(x)|\, d\lambda(x) \leq V_f(b).$$

Thm (The Fundamental Theorem of Calculus)

$f:[a,b]\to\mathbb{R}$ Absolutely Continuous $\iff$ f' exists a.e, $f'\in L^1([a,b])$ and $f(x)-f(a)=\int_a^x f'(t)\,d\lambda(t)$, $\forall x\in[a,b]$

Proof:

Let $f\in AC([a,b])$

Then $f\in BV([a,b])$

Hence, f' exists a.e. From previous Theorem we have:

$$\int_a^b |f'(x)|\,d\lambda(x)\le V_f(b)$$

Hence $f'\in L^1([a,b])$.

We define:

$$F(x)=\int_a^x f'(t)\,d\lambda(t)$$

Since f' is integrable, the first part of the fundamental Theorem of Calculus yields:

$$\boxed{F'(x) = f'(x) \quad \text{for } \lambda\text{-a.e. } x \in [a,b]}$$

We now consider the function:

$$F - f.$$

We have:

$$\boxed{(F-f)'(x) = 0 \quad \text{for } \lambda\text{-a.e. } x \in [a,b]}$$

We proved in a previous theorem that if the derivative of an absolute continuous function is zero λ-a.e., then that function must be constant. Since $f \in AC([a,b])$ we now proceed to show:

Claim: F is A.C.

Proof of Claim: Let $\{[a_i, b_i]\}_{i=1}^{K}$ be a non-overlapping collection of intervals in $[a,b]$. Then:

$$F(b_i) - F(a_i) = \int_{a_i}^{b_i} f'(t)\, d\lambda(t).$$

$$\Rightarrow \; |F(b_i) - F(a_i)| \leq \int_a^b |f'(t)|\, d\lambda(t)$$

thus $$\boxed{\sum_{i=1}^{K} |F(b_i) - F(a_i)| \leq \int_{\bigcup [a_i, b_i]} |f'|\, d\lambda} \rightarrow (6)$$

Define the measure:

$$\mu(E) = \int_E |f'| \, d\lambda \,, \quad \forall E \subset [a,b] \,,$$

E Lebesgue measurable

Then μ is a measure and:

$$\mu << \lambda$$

Thus, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$\boxed{\lambda(E) < \delta \implies \mu(E) < \varepsilon \quad \to (7)}$$

Indeed, recall how we can prove (7):
Proceed by contradiction a suppose $\exists \varepsilon > 0$ and a sequence of measurable sets $\{E_k\}$ such that:

$$\lambda(E_k) < \frac{1}{2^k} \quad \text{and} \quad \mu(E_k) > \varepsilon \,, \quad \forall k$$

We apply Borel Cantelli and define:

$$F := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$$

Then:

$$\lambda(F) = \lim_{m \to \infty} \lambda\left(\bigcup_{k=m}^{\infty} E_k\right)$$

$$\leq \lim_{m \to \infty} \left(\sum_{k=m}^{\infty} \frac{1}{2^k}\right) = 0$$

On the other hand:

$$\mu(F) = \lim_{m\to\infty} \mu\left(\bigcup_{k=m}^{\infty} E_k\right) \geq \varepsilon,$$

which contradicts that $\mu \ll \lambda$.

We now use (6) and (7) to obtain:

$$\sum_{i=1}^{k} |b_i - a_i| < \delta \implies \mu\left(\bigcup [a_i, b_i]\right) = \int_{\bigcup [a_i,b_i]} |f'| \, d\lambda \leq \varepsilon$$

and

$$\sum_{i=1}^{k} |F(b_i) - F(a_i)| < \varepsilon.$$

We completed the proof of $F \in AC([a,b])$.

Therefore we had before:

$(F-f)'(x) = 0$ for λ-a.e. $x \in [a,b]$, and

Since now we have $F - f \in AC([a,b])$

we conclude:

$$\boxed{F - f = \text{Constant}}$$

$$\therefore \boxed{F(x) - f(x) = F(a) - f(a), \quad \forall x \in [a,b]}$$

Since $F(a) = 0$ then $F(x) = f(x) - f(a)$, or:

$$\boxed{f(x) - f(a) = \int_a^x f'(t) \, d\lambda(t), \quad \forall x \in [a,b]}$$

We now prove the converse of the Fundamental Theorem of Calculus:

Assume f' exists a.e., $f' \in L^1([a,b])$ and:

$$f(x) - f(a) = \int_a^x f'(t)\, d\lambda(t), \quad \forall x \in [a,b].$$

We define:

$$F(x) = \int_a^x f'(t)\, d\lambda(t)$$

Then we proved earlier that

$$F \in AC([a,b])$$

$$\therefore\ f(x) - f(a) \in AC([a,b])$$

$$\therefore\ f \in AC([a,b]). \quad \blacksquare$$