Thm: Let $f \in L^1_{loc}(a,b)$. Then

$g = f$ λ-almost everywhere, for some $g \in BV[a,b]$ $\iff$ $f' = \mu$, $|\mu|(a,b) < \infty$ (f' is the distributional derivative of f)

Proof: $\Rightarrow$ Assume $f = g$ λ-almost everywhere and $g \in BV([a,b])$. Then

$g = g_1 - g_2$, g_1, g_2 non-decreasing (WLOG assume g_1, g_2 right continuous)

Let T_1 be the distribution corresponding to g_1. Thus:

$$T_1(\varphi) = \int_a^b g_1(x)\varphi(x)\,d\lambda(x)$$

and:

$$T_1'(\varphi) = -T_1(\varphi')$$

$$= -\int_a^b g_1(x)\varphi'(x)\,\underline{d\lambda(x)} \quad ; \text{ Lebesgue integral}$$

$$= -\int_a^b g_1(x)\varphi'(x)\,\underline{dx} \quad ; \text{ Riemann integral.}$$

$$= -\int_a^b g_1(x)\,\underline{d\varphi} \quad ; \text{ Riemann-Stieltjes integral; exercises 6.17, 6.18, 6.19 (Ziemer-Torres book).}$$

$\therefore$

$$T_1'(\varphi) = -\int_a^b g_1(x)\, d\varphi;$$

$d\varphi = \varphi' dx$
Riemann-Stieltjes integral.

$$= \int_a^b \varphi \, dg_1 \;;$$

Exercises 6.20
Integration by parts for Riemann-Stieltjes integrals.

$$= \int_a^b \varphi(x) g_1'(x)\, dx \;;$$

$dg_1 = g_1' \, dx$.
See for example Rudin's book, Chapter 6, Theorem 6.17

$$= \int_a^b \varphi(x)\, g_1'(x)\, d\lambda(x)$$

$$= \int_a^b \varphi(x)\, d\lambda g_1 \;;$$

excersice 6.21:
Riemann-Stieltjes and Lebesgue-Stieltjes integrals are in agreement.

(A) $\therefore$ $\boxed{T_1'(\varphi) = \int_a^b \varphi \, d\lambda_{g_1}}$

In the same way:

$$\boxed{T_2'(\varphi) = \int_a^b \varphi \, d\lambda_{g_2}, \quad \varphi \in \mathcal{D}(a,b)}.$$

Hence:

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If T_g is the distribution corresponding to g; since $T_g = T_1 - T_2$ then:

$$\boxed{T_g' = T_1' - T_2' = \lambda_{g_1} - \lambda_{g_2}; \text{ by (A)}}$$

Note: If g_1, g_2 are not right continuous, then by Problem 7.10 (see also Theorem 61.2 and Problem 3.62), we can redefine g_1, g_2 to $\tilde{g}_1, \tilde{g}_2$, such that $g_1 = \tilde{g}_1$, $g_2 = \tilde{g}_2$ almost everywhere and:

$\tilde{g}_1, \tilde{g}_2$ are non-decreasing, and right continuous.

Then:

$\tilde{g} := \tilde{g}_1 - \tilde{g}_2$ is of bounded variation (since any non-decreasing function is of bounded variation, $\tilde{g}_1, \tilde{g}_2$ are BV and hence $\tilde{g}$ is BV).

$\therefore$ $f = g = \tilde{g}$ almost everywhere, with $g, \tilde{g} \in BV$.

$$T_f = T_g = T_{\tilde{g}} = \lambda_{g_1} - \lambda_{g_2} = \lambda_{\tilde{g}_1} - \lambda_{\tilde{g}_2}$$

$\Leftarrow$ Suppose now that:

$$f' = \mu \quad ; \quad \text{i.e.}$$

$$-\int_a^b f\, \varphi'\, d\lambda = \int_a^b \varphi\, d\mu, \quad \forall \varphi \in \mathcal{D}(a,b),$$

where:

μ is a signed measure $|\mu|(a,b) < \infty$

Then:

$$\mu = \mu_1 - \mu_2, \qquad \mu_1, \mu_2 \text{ non-negative measures}$$

Define:

$$f_i(x) = \mu_i((-\infty, x]), \quad i = 1, 2$$

(extend μ_i by $\mu_i(\mathbb{R} \setminus (a,b)) = 0$).
(f_i are non-decreasing and right continuous)

$\therefore$ Theorem 96.1 $\Rightarrow$

$$\mu_i = \lambda_{f_i}, \qquad \lambda_{f_i} \text{ is the Lebesgue-Stieltjes measure}$$

(on all Borel sets)

$$\therefore T'_{f_i}(\varphi) = -T_{f_i}(\varphi') = -\int_a^b f_i \varphi'\, d\lambda = \int_a^b \varphi\, d\lambda_{f_i} = \int_a^b \varphi\, d\mu_i$$

Define

$$\boxed{g := f_1 - f_2}$$

g is BV and $T_g' = T_{f_1}' - T_{f_2}' = \mu_1 - \mu_2 = \mu$

Hence:

$$T_f' = \mu$$

$$T_g' = \mu$$

$$\therefore\ (T_f - T_g)' = 0$$

$$\therefore\ T_f = T_g + K$$

$$\therefore\ \int_a^b f\varphi\, d\lambda = \int_a^b g\varphi\, d\lambda + \int_a^b K\varphi\, d\lambda$$

$$\therefore\ \int_a^b (f - g - K)\varphi\, d\lambda = 0\,, \quad \forall \in \mathcal{D}(a,b)$$

$$\therefore\ f = g + K \ \lambda\text{-almost everywhere}$$

Since $g+K$ is BV we conclude the f is equivalent to a BV function.

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- **Note:** If

$$f = g \quad \lambda\text{-almost everywhere}$$

Then

$$T_f' = T_g',$$

where T_f, T_g are the distributions corresponding to f and g respectively.

Def: The essential variation of a function f defined on (a,b) is:

$$\text{ess } V_a^b f = \sup \left\{ \sum_{i=1}^{k} |f(t_{i+1}) - f(t_i)| \right\}$$

where the supremum is taken over all finite partitions $a < t_1 < \cdots < t_{k+1} < b$ such that each t_i is a point of approximate continuity of f.

Note: If

$$f = g \quad \lambda\text{-almost everywhere}$$

Then

$$\text{ess } V_a^b f = \text{ess } V_a^b g$$

<u>Def</u>: Let μ be a signed <u>finite</u> Radon measure in the open set Ω. The norm of μ, is defined as:

$$\|\mu\| := |\mu|(\Omega)$$

We have:

<u>Thm 1</u>: Suppose $f \in L^1(a,b)$. Then:

$f' = \mu$ (in the sense of distributions), $|\mu|(a,b) < \infty$ $\Longleftrightarrow$ $\operatorname{ess} V_a^b f < \infty$

Moreover:

$$\|f'\| = |\mu|(a,b) = \operatorname{ess} V_a^b f$$

The proof of this Theorem uses the fact that, if μ is a signed finite measure in Ω, then:

$$\boxed{|\mu|(\Omega) = \sup\left\{ \int_\Omega \varphi \, d\mu : \varphi \in C_c(\Omega), |\varphi| \le 1 \right\}} \quad (1)$$

(1) is clear from the proof of RRT3, Local & Global Version. However, we present again the proof of it next:

<u>Lemma 1</u>: Let μ be a (signed) finite Radon measure in a set $E \subset \mathbb{R}^n$. Then, for every open set $\Omega \subset E$:

$$|\mu|(\Omega) = \sup \left\{ \int_\Omega f \, d\mu : f \in C_c(\Omega), |f| \le 1 \right\}.$$

<u>Note:</u> If $E \subsetneq \mathbb{R}^n$, we consider E as a metric space endowed with the induced topology from $\mathbb{R}^n$. Thus, $\Omega \subset E$ is open in the relative-topology

<u>Proof:</u>

Since $\mu << |\mu|$, Radon-Nikodym Theorem yields:

$\exists\, f \in L^1(E)$ s.t. $\mu = f|\mu|$. That is:

$$\mu(A) = \int_A f \, d|\mu|, \quad A \subset E \text{ measurable.}$$

Then:
$$|\mu|(A) = \int_A |f| \, d|\mu|, \quad A \subset E$$

and this implies $|f(x)| = 1$, $|\mu|$-a.e. x.

We have therefore:

$$\boxed{\mu = f|\mu| \ , \quad |f| = 1 \quad \text{a.e. } x} \qquad (2)$$

Choose now a sequence $\{f_k\} \subset C_c(\Omega)$ such that:

$$f_k(x) \to f(x), \ |\mu|\text{-a.e. } x, \quad |f_k| \le 1.$$

(We can choose f_k by applying Lusin's Theorem to f).

Since $|\mu|(\Omega) < \infty$, we can apply the Lebesgue Dominated convergence Theorem:

$$\lim_{k\to\infty} \int_\Omega f_k \, d\mu = \lim_{k\to\infty} \int_\Omega f_k \cdot f \, d|\mu|; \text{ by (2)}$$

$$= \int_\Omega f \cdot f \, d|\mu|$$

$$= \int_\Omega f^2 \, d|\mu|$$

$$= \int_\Omega d|\mu|; \quad \text{by (2)}$$

$$= |\mu|(E)$$

Therefore; for every $\varepsilon > 0$, $\exists N(\varepsilon)$ s.t.

$$|\mu|(E) \le \int_\Omega f_k \, d\mu + \varepsilon, \quad \forall k \ge N(\varepsilon)$$

$$\le \sup\left\{ \int_\Omega f \, d\mu, \ f \in C_c(\Omega), \ |f| \le 1 \right\} + \varepsilon$$

$$\therefore \boxed{|\mu|(E) \le \sup\left\{ \int_\Omega f \, d\mu, \ f \in C_c(\Omega), \ |f| \le 1 \right\}} \quad (3)$$

(since ε is arbitrary)

For the reverse inequality:

$$\int_\Omega f\,d\mu \le \int_\Omega |f|\,d|\mu|$$
$$\le \|f\|_\infty |\mu|(\Omega)$$

$$\therefore \int_\Omega f\,d\mu \le |\mu|(\Omega), \quad f \in C_c(\Omega), \ |f| \le 1$$

Hence:

$$\boxed{\sup\left\{\int_\Omega f\,d\mu : f \in C_c(\Omega), \ |f| \le 1\right\} \le |\mu|(\Omega)} \quad (4).$$

Actually, notice that:

$|\mu|(\Omega) = \sup\left\{\int_\Omega f\,d\mu : f \in C_c^\infty(\Omega), \ |f| \le 1\right\}$,

Since $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$.

Also:

$$|\mu|(\Omega) = \sup\left\{\int_\Omega f\,d\mu : f \in C_0(\Omega), \ |f| \le 1\right\}$$
$$= \sup\left\{\int_\Omega f\,d\mu : f \in C(\Omega), \ |f| \le 1\right\}$$

In fact, for $C_0(\Omega)$ or $C(\Omega)$ we can prove (3) and (4) following the same proof as in Lemma 1, since:

$$|\mu|(E) \le \int_\Omega f_k\,d\mu + \varepsilon, \ \forall k \ge N(\varepsilon)$$
$$\le \sup\left\{\int_\Omega f\,d\mu, \ f \in C_0(\Omega), \ |f| \le 1\right\} + \varepsilon$$
$$\le \sup\left\{\int_\Omega f\,d\mu, \ f \in C(\Omega), \ |f| \le 1\right\} + \varepsilon$$

Alternative definitions of functions of bounded variation (BV) in one-dimension.

Let $f \in L^1(a,b)$. We say that $f \in BV(a,b)$ iff:

(a) $\operatorname{ess} V_a^b f < \infty$

or

(b) The distributional derivative of f is a finite Radon measure in (a,b)

or

(c) $\sup \left\{ \int_a^b f \varphi' \, d\lambda(x) : \varphi \in C_c^1(a,b), |\varphi| \leq 1 \right\} < \infty$

We can extend the definition (c) to several variables and define the space of functions of bounded variation in an open set $\Omega \subset \mathbb{R}^n$:

$$BV(\Omega).$$

BV functions in $\mathbb{R}^n$.

<u>Def</u>: A function $f \in L^1(\Omega)$, Ω open, has bounded variation in Ω if

$$\sup \left\{ \int_\Omega f \operatorname{div} \varphi \, d\lambda(x) : \varphi \in C_c^1(\Omega; \mathbb{R}^n), |\varphi| \leq 1 \right\} < \infty$$

We write

$$BV(\Omega)$$

to denote the space of functions of bounded variation.

<u>Def</u>: A function $f \in L^1_{loc}(\Omega)$ has locally bounded variation in Ω if for each open set $V \subset\subset \Omega$,

$$\sup \left\{ \int_V f \operatorname{div} \varphi \, d\lambda(x) : \varphi \in C_c^1(V; \mathbb{R}^n), |\varphi| \leq 1 \right\} < \infty.$$

We write

$$BV_{loc}(\Omega)$$

to denote the space of such functions.

Thm : Structure theorem for BV_{loc} functions:

Let $f \in BV_{loc}(\Omega)$. Then there exists a Radon measure μ on Ω and a μ-measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that:

(i) $|\sigma(x)| = 1$ μ-a.e. , and

(ii) $\int_\Omega f \operatorname{div} \varphi \, d\lambda(x) = -\int_\Omega \varphi \cdot \sigma \, d\mu$

Moreover, (ii) asserts that the weak first partial derivatives of a BV function are Radon measures.

Proof:

Use the RRT3 , local version, to prove (i) and (ii). Then show that (ii) implies that

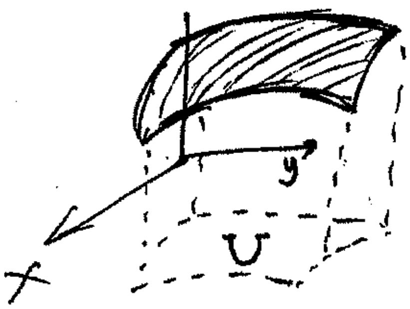
$$\frac{\partial f}{\partial x_i} = \tilde{\mu}_i \quad , \quad i = 1, \dots, n,$$

where $|\tilde{\mu}_i|(V) < \infty$, for each $V \subset\subset \Omega$.

Complete the details of this proof from Evans' book: "Fine properties of functions..."

Ex 3: Surface area of a graph

Let $g: \mathbb{R}^2 \to \mathbb{R}$ Lipschitz.



Define:

$$f: \mathbb{R}^2 \to \mathbb{R}^3$$
$$f(x,y) = (x, y, g(x,y)), \quad f \text{ is } 1\text{-}1 \text{ on } U$$

Then:

$$df(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{3\times 2}$$

The Binet-Cauchy Formula (Theorem 4, "Measure Theory and fine properties of functions) implies:

$$(Jf)^2 = \text{sum of squares of } (2\times 2)\text{-subdeterminants}$$

$$= 1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2$$

$\therefore$

$$|Jf(x,y)| = \sqrt{1 + |\nabla g(x,y)|^2}$$

From Area Formula:

$$\int_{g(U)} d\mathcal{H}^2 = \int_U |Jf(x,y)|\, dx\, dy$$

$$= \int_U \sqrt{1+|\nabla g|^2}$$

$$\therefore \boxed{\mathcal{H}^2(U) = \text{Area of } U = \int_U \sqrt{1+|\nabla g|^2}}$$

Recall the Plateau Problem:

Minimize $\int_U \sqrt{1+|\nabla g|^2}$, over all function g such that $g = \gamma$ on ∂U.

If g minimizes area, then g satisfies the minimal surface equation:

$$\boxed{\operatorname{div}\left(\frac{\nabla g}{(1+|\nabla g|^2)}\right) = 0. \quad (*)}$$

Indeed, Let $\alpha(t) = \int_U \sqrt{1+|\nabla(g+t\varphi)|^2}\, dx\, dy$

$$\therefore \alpha'(t) = \int_U \frac{1}{2}\left(1+|\nabla(g+t\varphi)|^2\right)^{-1/2} \cdot \frac{d}{dt}\left(1+(g_x+t\varphi_x)^2+(g_y+t\varphi_y)^2\right)$$

$$\therefore \alpha'(t) = \int_U \frac{1}{2}\left(1+|\nabla(g+t\varphi)|^2\right)^{-1/2}\left[2(g_x+t\varphi_x)\varphi_x + 2(g_y+t\varphi_y)\varphi_y\right]$$

$$\Rightarrow \alpha'(0) = \int_U (1+|\nabla g|^2)^{-1/2}(\nabla g \cdot \nabla\varphi)\, dx\, dy \Rightarrow \int_U \operatorname{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right)\varphi = 0 \ \forall \varphi$$

$$\Rightarrow (*)$$